WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

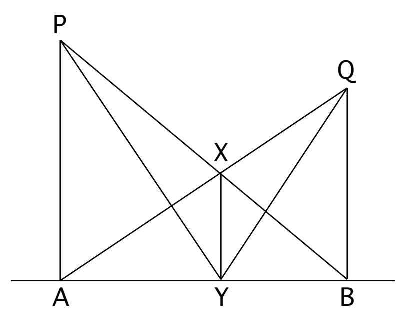
SOLUTIONS TO PROBLEM SET II (2010-2011)

1. Twenty lines are drawn in the plane with no pair of them parallel. Every two of the lines thus intersect at a point, and we suppose that there are exactly 174 points that lie on more than one of our lines. Prove that some point lies on exactly three of the given lines.

SOLUTION. The number of ways of choosing two objects out of a set of n is $n(n-1)/2$. In particular, the number of ways of choosing two lines out of the given twenty is $(20)(19)/2 = 190$. Since every two of the lines meet in one point, we might expect there to be 190 points lying on more than one line, but there are only 174, so there are 16 "missing" intersection points. This happens, of course, because there are some points that lie on three or more lines.

Let us say that a point is a k-point if exactly k of the given lines go through it. The k lines through a k-point have only one intersection point rather than the $k(k-1)/2$ intersection points that might be expected, so each k-point accounts for $k(k-1)/2 - 1$ of the 16 missing points. A 3-point, therefore, accounts for 2 missing points; a 4-point accounts for 5 missing points; a 5-point accounts for 9, and a 6-point accounts for 14 missing points. A k-point with $k \geq 7$ would yield at least 20 missing points, which is too many. Thus 16 can be written as a sum of numbers from the set $\{2, 5, 9, 14\}$, corresponding to 3-points, 4-points, 5-points and 6-points. Our goal is to show that at least one of the summands must be 2, which will establish the existence of a 3-point. If 14 is one of the summands, the only possibility is $14 + 2 = 16$, so 2 occurs, as wanted. If 9 occurs as a summand, it is easy to see that the only possibility is $9 + 5 + 2 = 16$, and again 2 occurs. We can now assume that neither 9 nor 14 is a summand, and since 16 is not a multiple of 5, it cannot be that 5 is the only summand. In all cases, therefore, 2 occurs, and the proof is complete.

2. In the diagram, lines $\overline{PA}$ and $\overline{QB}$ are perpendicular to line $\overline{AB}$. Point X is the intersection of $\overline{PB}$ and $\overline{QA}$, and the perpendicular $\overline{XY}$ is dropped from X to $\overline{AB}$. Prove that $\overline{XY}$ bisects $\angle PYQ$.



SOLUTION. Since $\overline{XY}$ is parallel to $\overline{PA}$, it follows that $\triangle PAB$ is similar to $\triangle XYB$. Then $PA/XY = AB/YB$, and we have $(PA)(YB) = (AB)(XY)$. Similarly, $(QB)(YA) = (AB)(XY)$. It follows that $(PA)(YB) = (QB)(YA)$, and this yields $PA/QB = YA/YB$. Now consider $\triangle PAY$ and $\triangle QBY$. In these triangles, $\angle A = 90° = \angle B$, and we have seen that the sides including these equal angles are in proportion. It follows by the SAS similarity criterion that $\triangle PAY$ and $\triangle QBY$ are similar, and thus $\angle PYA = \angle QYB$. Since $\overline{XY}$ is perpendicular to $\overline{AB}$, we see that $\angle PYX$ and $\angle QYX$ are complementary to these equal angles, and thus $\angle PYX = \angle QYX$, as wanted.

3. Let f and g be real-valued functions defined on the real numbers, and let the sets A and B be given by $A = \{x \mid f(g(x)) = x\}$ and $B = \{x \mid g(f(x)) = x\}$. Assuming that A and B are finite (but possibly empty) sets, show that A and B contain equal numbers of elements.

SOLUTION. Let a be a member of set A, and let $b = g(a)$. Then $f(g(a)) = a$, so $g(f(b)) = g(f(g(a))) = g(a) = b$, and thus b lies in set B. Next, we show that if a_1 and a_2 are *different* members of A, then $g(a_1)$ and $g(a_2)$ must be different members of B. We already know that $g(a_1)$ and $g(a_2)$ lie in B, so suppose that $g(a_1) = g(a_2)$. Then $a_1 = f(g(a_1)) = f(g(a_2)) = a_2$, where the first and third equalities hold because a_1 and a_2 lie in A. This contradicts the assumption that

$a_1 \neq a_2$, and thus $g(a_1)$ and $g(a_2)$ really are different. We now know that the function g takes different members of A to different members of B, and it follows that the number of members of A is at most equal to the number of members of B. Reasoning similarly with the roles of A and B reversed, we conclude that the number of members of B is at most the number of members of A. Combining these two inequalities, we see that A and B have equal numbers of members.

4. Find all integers n with the property that

$$\sqrt{n+12\sqrt{5}} \;-\; \sqrt{n-12\sqrt{5}}$$

is also an integer.

SOLUTION. Suppose $\sqrt{n+12\sqrt{5}} - \sqrt{n-12\sqrt{5}} = k$, where k is a (necessarily positive) integer. Bringing the second square root to the right and squaring both sides yields

$$n+12\sqrt{5} = k^2 + 2k\sqrt{n-12\sqrt{5}} + (n-12\sqrt{5})\,.$$

Algebraic manipulation now yields $2k\sqrt{n-12\sqrt{5}} = 24\sqrt{5} - k^2$. Squaring both sides, we obtain $4k^2(n-12\sqrt{5}) = k^4 - 48k^2\sqrt{5} + 5(24)^2$, and simplifying, we get $4k^2n = k^4 + 5(24)^2$. It follows that k^2 must be a divisor of $5(24)^2$, and thus k^2 divides $(24)^2$, so k is a divisor of 24, and we can write $24 = km$ for some integer m. We thus have $4k^2n = k^4 + 5k^2m^2$, and therefore $4n = k^2 + 5m^2$.

Since $km = 24$, at least one of k or m must be even. But $k^2 + 5m^2 = 4n$ is a multiple of 4, so it is not possible that only one of k or m is even, and hence they are both even. Thus k is an even divisor of 24 with $24/k$ also even, and it follows that k is one of 2, 4, 6 or 12. The corresponding values of n are 181, 49. 29 and 41, respectively, so these are the only possible values of n. But $k = 12$ and $n = 41$ is not a solution to our original problem because $\sqrt{41+12\sqrt{5}} < 12$.

We must check that 181, 49 and 29 actually are solutions. We might guess that $\sqrt{n+12\sqrt{5}}$ has the form $a + b\sqrt{5}$. This yields $2ab = 12$, and there are only a few possibilities. Trying $a = 1$, $b = 6$, for example, we discover that $(1+6\sqrt{5})^2 = 181 + 12\sqrt{5}$. Now $(1-6\sqrt{5})^2 = 181 - 12\sqrt{5}$, and since $1 - 6\sqrt{5} < 0$, it follows that $\sqrt{181-12\sqrt{5}} = 6\sqrt{5} - 1$. We conclude easily that $k = 2$ and thus $n = 181$ really is a solution. Similarly we see that $n = 49$ and $n = 29$ are also solutions.

5. At noon, the hour hand, minute hand and second hand of an accurately set clock all point in the same direction. Find the next time that the three hands coincide, and prove that your answer is correct.

SOLUTION. During the 12 hour period from noon to midnight, the hour hand makes one revolution around the clock face, and the minute hand makes 12, so the minute hand overtakes the hour hand 11 times in 12 hours. Since these "passings" are clearly equally spaced, it follows that the interval between consecutive passings is 12/11 hours. In the hour it takes for the minute hand to make one revolution, the second hand makes 60, so the second hand overtakes the minute hand 59 times in one hour, and thus the interval between these passings is 1/59 hours. Now suppose that the first time after noon when the three hands coincide happens t hours after noon. Then the real number t must be an integer multiple of 12/11, and it must also be an integer multiple of 1/59. It follows that there exist positive integers m and n such that $m(12/11) = t = n/59$. Then $(12)(59)m = 11n$, and since 11 is a prime dividing neither 12 nor 59, we deduce that 11 divides m, and we can write $m = 11k$ for some positive integer k. Then $t = 12m/11 = 12k$ and since, $k \geq 1$, we have $t \geq 12$. Thus the first time after noon when the three hands coincide is midnight.