WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2010-2011)

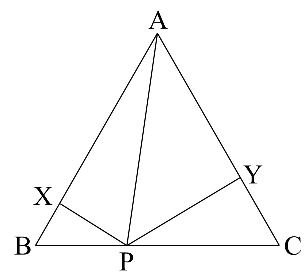
1. Find all positive integer solutions of $x^x + y^y + z^z = 3xyz$.

SOLUTION. We can assume that $x \le y \le z$. If $z \ge 4$, then $3xyz = x^x + y^y + z^z \ge z^z \ge z^4 \ge 4z^3$, so canceling z from both sides yields $3xy > 4z^2 \ge 4xy$, where the last inequality holds because $z \ge x$ and $z \ge y$. This is a contradiction, and we conclude that $z \le 3$.

If $z = 3$, our equation becomes $x^x + y^y + 27 = 9xy$, so $9xy > 27$, and thus $xy > 3$. On the other hand, x and y are at most $z = 3$, so x and y must be at least 2. If $y = 3$, we have $x^x + 27 + 27 = 27x$, so $x = 3$ yields a solution and $x = 2$ does not. The remaining possibility is $x = 2 = y$, and checking, we see that this is not a solution. The only solution with $z = 3$, therefore, is $(x, y, z) = (3, 3, 3)$.

If $z = 2$, we have $x^x + y^y + 4 = 6xy$ and each of x and y is 1 or 2. Since $x^x + y^y = 6xy - 4$ is even, either x and y are both 1 or x and y are both 2. If $x = 2 = y$, we get $4 + 4 + 4 = 24$, which is false, but if $x = 1 = y$, our equation becomes $1 + 1 + 4 = 6$, which is true. Thus $(x, y, z) = (1, 1, 2)$ is a solution, and it is the only solution with $z = 2$. Finally, if $z = 1$, we must have $x = 1 = y$, and we check that this is a solution. We have now found three solutions: $(3, 3, 3)$, $(1, 1, 2)$ and $(1, 1, 1)$ under the assumption that $x \le y \le z$. Without this assumption, we must consider rearrangements, and we get two more solutions, namely $(1, 2, 1)$ and $(2, 1, 1)$.

2. Suppose that $\triangle ABC$ is isosceles with base $\overline{BC}$. A point P on $\overline{BC}$ is chosen, and perpendiculars $\overline{PX}$ and $\overline{PY}$ are dropped from P to $\overline{AB}$ and $\overline{AC}$ (possibly extended). Show that the total length $PX + PY$ remains constant as P moves along side $\overline{BC}$.



SOLUTION. Since the triangle is isosceles, we can let $s = AB = AC$. Draw $\overline{PA}$ and observe that the area of $\triangle PAB$ is $(1/2)PX \cdot AB = (s/2)PX$, and similarly, the area of $\triangle PAC$ is $(s/2)PY$. The area K of the original triangle is the sum of these two areas, so we have $K = (s/2)PX + (s/2)PY = (s/2)(PX + PY)$. Thus $PX + PY = 2K/s$, and this does not depend on the point P.

3. Twenty closed fruit crates are lined up on the floor. Ten of the crates are labeled "APPLES" and the other ten are labeled "PEARS". I want to select at least one crate of apples and one of pears from among the crates. This would be easy, except that some of the crates have been incorrectly labeled, and I do not know which are the mislabeled crates or how many of them there are. I do know, however, that of every five consecutive crates, at most two are wrongly labeled. Find a strategy that allows me to choose six crates in such a way that I can be sure that at least one crate of apples and one of pears is among my six. Also, prove that there is no strategy guaranteed to accomplish this when only five crates are selected.

SOLUTION. Let us imagine that the numbers 1 through 20 are painted on the crates in the order they are lying on the floor. At least three of the labels on crates 1 through 5 must be identical, so we can assume that the first five crates include at least three crates labeled "APPLES". Take these three. Since at most two of crates 1 through 5 are mislabeled, we can be sure that at least one of the three crates we have taken is correctly labeled, and so it contains apples. Since there are equal numbers of apple and pear labels, it is not possible that each of the sets of crates 1–5, 6–10, 11–15 and 16–20 bear a majority of apple labels, so one of these sets of five crates contains at least three crates labeled "PEARS". Take these three, so now we have chosen a total of six crates, and we are sure that among them is at least one crate of apples. At most two of our three "PEAR" crates are mislabeled, so we are guaranteed to have at least one crate of pears.

To see that no successful strategy exists if we choose only five crates, observe that as far as we know, any two of the twenty crates may be mislabeled. Suppose a strategy for selecting five crates exists such that at least one crate of apples and one of pears is guaranteed to be among them. Select five crates using this strategy, and observe that either the label "APPLES" or the label "PEARS" occurs two or fewer times. Suppose, for example, that of our five selected crates, at most two are labeled "APPLES". It is possible that all twenty of the crates are correctly labeled except for the crates labeled "APPLES" in our chosen set of five. If that happens, all five of our crates contain pears and our supposed strategy fails.

4. I have 64 numbered tiles which contain 64 different numbers. I want to place these tiles into the 64 boxes of an 8×8 chessboard. There are of course, a huge number of ways to do this. Let us say that a placement of tiles into boxes is "good" if the sums of the tiles in all eight rows are equal, and that the placement is "very bad" if no two of the row sums are equal. Prove that there cannot be more good placements than there are very bad placements.

SOLUTION. Given a good placement, we will construct from it a very bad placement, and we will show that different good placements will yield different very bad placements. In addition, there may be some very bad placements that cannot be constructed from good placements, so the total number of very bad placements is at least equal to the number of good placements.

Suppose we have a particular good placement. Let $b_1, b_2, \ldots, b_8$ be the eight numbers in the last column of this placement, and note that these numbers b_i are all different. Do not number the b's so that b_i is in the ith row, but instead, number them so that the b_i are in increasing order, that is $b_1 < b_2 < \cdots < b_8$. Let s_i be the sum of the first seven tiles in whichever row ends with b_i. Since the placement is good, we know that the eight quantities $s_i + b_i$ are all equal. It follows that $s_1 > s_2 > \cdots > s_8$.

Now we construct a new placement by swapping b_1 and b_8, swapping b_2 and b_7, swapping b_3 and b_6 and swapping b_4 and b_5. The row that formerly ended with b_1 now ends with b_8, so its sum is $s_1 + b_8$. The row that originally ended with b_2 now has row sum $s_2 + b_7$, and in general, the new row sums are $s_i + b_{9-i}$. Now $s_1 + b_8 > s_2 + b_7$ since $s_1 > s_2$ and $b_8 > b_7$. Similarly, $s_2 + b_7 > s_3 + b_6$, and so on. Thus all the new row sums are different, and the new placement is very bad. Given this very bad placement, we can see what the original good placement was by simply reversing the order of the last tiles. This completes the proof.

5. Let $a > 0$ and $b > 0$. Prove that

$$\frac{a}{a+b^4} + \frac{b}{b+a^4} > \frac{1}{1+a^2b^2}.$$

SOLUTION. The inequality holds if either one of the two fractions on the left exceeds the fraction on the right. In particular, we are done if $a/(a+b^4) > 1/(1+a^2b^2)$. Since $a/(a+b^4) = 1/(1+b^4/a)$, we are done if $b^4/a < a^2b^2$, or equivalently, $b^2 < a^3$. We can assume, therefore, that $b^2 \geq a^3$, and similarly, we can assume that $a^2 \geq b^3$. Then $a^4 \geq b^6 \geq a^9$, and it follows that $a \leq 1$, and similarly, $b \leq 1$. Then $b^4 \leq b$, so $a/(a+b^4) \geq a/(a+b)$. Similarly, $b/(b+a^4) \geq b/(b+a)$, so the left side of the inequality we are trying to prove is at least $(a+b)/(a+b) = 1$. On the other hand, the right side is less than 1, and this establishes the inequality.