1. Find a simple formula for the sum \(2 \cdot 4 + 3 \cdot 8 + 4 \cdot 16 + 5 \cdot 32 + \cdots + n \cdot 2^n\), and prove that your formula is correct.

**SOLUTION.** Let us write \(S_n\) to denote the sum of the \(n - 1\) numbers in the given list. For \(n = 2, 3, 4, 5, 6\), we compute that \(S_n = 8, 32, 96, 256, 640\), and we observe that these sums can be written as \(1 \cdot 2^3, 2 \cdot 2^3, 3 \cdot 2^5, 4 \cdot 2^6, 5 \cdot 2^7\). This suggests that the general formula should be \(S_n = (n - 1)2^{n+1}\). This formula can be proved by induction, but perhaps the following is a more elegant proof. Starting with the definition of \(S_n\) and doubling it, we have

\[
S_n = 2 \cdot 4 + 3 \cdot 8 + 4 \cdot 16 + 5 \cdot 32 + \cdots + n \cdot 2^n \quad \text{and} \quad 2S_n = 2 \cdot 8 + 3 \cdot 16 + 4 \cdot 32 + \cdots + (n - 1) \cdot 2^n + n \cdot 2^{n+1}.
\]

Subtracting the second equation from the first, we get

\[
-S_n = 2 \cdot 4 + 8 + 16 + 32 + \cdots + 2^n - n \cdot 2^{n+1}.
\]

It is easy to see that \(4 + 4 + 8 + 16 + \cdots + 2^n = 2^{n+1}\), and this yields \(-S_n = 2^{n+1} - n \cdot 2^{n+1}\), and thus \(S_n = (n - 1)2^{n+1}\), as wanted.

2. In the diagram, \(P\) is a point on side \(BC\) of equilateral \(\triangle ABC\). Lines \(PQ\) and \(PR\) are drawn parallel to \(AC\) and \(AB\), respectively, where \(Q\) lies on \(AB\) and \(R\) lies on \(AC\), and then \(BR\) and \(CQ\) are drawn. Prove that \(BR = CQ\).

**SOLUTION.** Observe that \(\triangle BQP\) is equilateral since all of its angles are \(60^\circ\), and thus \(QP = BP\). Similarly, \(\triangle PRC\) is equilateral, and we have \(RP = PC\). Also, we see that \(\angle BPR = 180^\circ - \angle RPC = 120^\circ\), and \(\angle QPC = 180^\circ - \angle QPB = 120^\circ\). It follows that \(\triangle RPB \cong \triangle CPQ\) by side-angle-side, and we conclude that \(BR = CQ\) as wanted, because \(BR\) and \(CQ\) are corresponding parts of these congruent triangles.

3. Let \(n \geq 1\) be an integer and let \(t\) denote the number of positive integer divisors of \(n^2\). Show that the number of positive integer solutions \((a, b)\) of the equation \(1/a - 1/b = 1/n\) is precisely equal to \((t - 1)/2\).

**SOLUTION.** Given an arbitrary positive integer \(n\), we count the number of positive integer solutions for the equation \(1/a - 1/b = 1/n\). Given a solution \((a, b)\), we have \(1/a = 1/n + 1/b > 1/n\), so \(a < n\) and we can write \(a = n - x\) for some integer \(x\), where \(0 < x < n\). Then \(1/b = 1/a - 1/n = 1/(n - x) - 1/n = x/(n(n - x))\), and we have \(b = n(n - x)/x = n^2/x - n\). To count solutions, therefore, we must count positive integers \(x < n\) such that the quantity \(n^2/x\) is an integer. In other words, we must count positive divisors of \(n^2\) that are less than \(n\).

Now we are given that \(t\) is the total numbers of divisors of \(n^2\). One of these \(t\) divisors is \(n\), and the remaining \(t - 1\) divisors can be paired by combining the divisor \(d\) with the divisor \(n^2/d\). Exactly one member of each of these \((t - 1)/2\) pairs is less than \(n\), so the number of positive divisors of \(n^2\) that are less than \(n\) is \((t - 1)/2\), where \(t\) is the total number of positive divisors of \(n^2\). It follows that the number of positive solutions to the equation \(1/a - 1/b = 1/n\) is exactly \((t - 1)/2\), as required.
4. (The new year’s problem.) Find the smallest positive integer \( n \) with the property that the equation \( \frac{1}{a} - \frac{1}{b} = \frac{1}{n} \) has exactly 2010 different solutions in positive integers \( a \) and \( b \).

**SOLUTION.** From the preceding problem, if \( t \) denotes the number of positive integer divisors of \( n^2 \), then we know that the number of positive integer solutions of the equation \( 1/a - 1/b = 1/n \) is precisely \( (t - 1)/2 \).

Now to compute \( t \), write \( n = p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r} \), where the \( p_i \) are different prime numbers and the exponents \( e_i \) are positive integers. Then \( n^2 = p_1^{2e_1}p_2^{2e_2} \cdots p_r^{2e_r} \), and the positive divisors of \( n^2 \) are the numbers \( m \) of the form \( m = p_1^{f_1}p_2^{f_2} \cdots p_r^{f_r} \), where each exponent \( f_i \) satisfies \( 0 \leq f_i \leq 2e_i \). In other words, there are \( 2e_i + 1 \) possibilities for the exponent \( f_i \), and we conclude that \( t = (2e_1 + 1)(2e_2 + 1)(2e_3 + 1) \cdots (2e_r + 1) \).

In the given problem, we are told that there are exactly 2010 positive solutions to the equation \( 1/a - 1/b = 1/n \), and thus \( (t - 1)/2 = 2010 \), and we have \( t = 4021 \). Now 4021 happens to be a prime number, and so the equation \( 4021 = t = (2e_1 + 1)(2e_2 + 1)(2e_3 + 1) \cdots (2e_r + 1) \) forces the number \( r \) of factors to be 1, and \( (2e_1 + 1) = 4021 \). Then \( e_1 = 2010 \), and the number \( n \) must have the form \( n = p^{2010} \), where \( p \) is a prime number. Also, this analysis shows that for every number \( p^{2010} \), the number of solutions is 2010. Since \( p = 2 \) is the smallest prime, the smallest positive integer \( n \) such that the equation \( 1/a - 1/b = 1/n \) has exactly 2010 positive solutions is therefore \( 2^{2010} \).

5. For any two integers \( x \) and \( y \), we write \( x \square y \) to denote a certain integer that is determined by \( x \) and \( y \). Suppose that the “\( \square \)” operation satisfies the following axioms.

(a) \( (x \square y) + (y \square z) + (z \square x) = 0 \) for all \( x, y, z \).

(b) \( z(x \square y) = (z x) \square (zy) \) for all \( x, y, z \).

(c) There exist integers \( x, y \) with \( x > y \) and such that \( x \square y = 1 \).

Compute \( 2010 \square 10 \).

**SOLUTION.** Setting \( x = y = z \), Axiom (a) yields \( 3(x \square x) = 0 \), and thus \( x \square x = 0 \) for all \( x \). Now setting \( z = y \) in Axiom (a) yields \( (x \square y) + (y \square y) + (y \square x) = 0 \), and since the middle term equals 0, we conclude that \( y \square x = -(x \square y) \) for all \( x \) and \( y \). Now set \( x = 1 \) and \( y = 0 \) in Axiom (b) to get \( z(1 \square 0) = z \square 0 \) for all \( z \), and write \( k = 1 \square 0 \). Then \( k \) is an integer and \( z \square 0 = k z \) for all \( z \). Now set \( z = 0 \) in Axiom (a), yielding \( 0 = (x \square y) + (y \square 0) + (0 \square x) = (x \square y) + ky - kx \), and thus \( x \square y = kx - ky = k(x - y) \) for all \( x \) and \( y \). By Axiom (c), we can choose \( x > y \) such that \( x \square y = 1 \). This yields \( 1 = k(x - y) \), and hence \( k > 0 \). Furthermore, since \( k \) and \( (x - y) \) are integers, we deduce that \( k = 1 \). We now have \( x \square y = k(x - y) = x - y \) for all \( x \) and \( y \), and thus \( 2010 \square 10 = 2000 \).