1. Given a positive integer $N$, write $N^*$ to denote the integer obtained by adding $N$ and all the digits of $N$. Thus $5^* = 10$ and $86^* = 100$. Also, $977^* = 1,000$ and $9968^* = 10,000$. Find an integer $N$ such that $N^* = 1,000,000$, or prove that no such integer exists.

**SOLUTION.** Suppose that $N^* = 1,000,000$. Then $N$ has at most six digits, so its digit sum is at most 54. Thus $N \geq 1,000,000 - 54 = 999,946$, and we can write $N = 999,9ab$, where $a$ and $b$ are digits. Then $1,000,000 = N^* = (999,900 + 10a + b) + (36 + a + b)$, and this yields $11a + 2b = 64$. Since $b \leq 9$, we have $11a \geq 64 - 18 = 46$, and since $b \geq 0$, we have $11a \leq 64$. Thus $46/11 \leq a \leq 64/11$, and since $a$ is an integer, we must have $a = 5$. Then $2b = 64 - 55 = 9$, and this is a contradiction since $b$ must be an integer. Therefore, we conclude that there is no integer $N$ such that $N^* = 1,000,000$.

2. In the figure, lines $AX$ and $AY$ are perpendicular tangents to a circle. Also, $BX$ and $BY$ are perpendicular tangents to the same circle. Prove that line $XY$ goes through the center of the circle.

**SOLUTION.** Let $R$, $S$, $T$ and $U$ be the four points of tangency, and let $r$, $s$, $t$ and $u$ be the lengths of the sides of the quadrilateral $XAYB$. Then $r = XR + AR$, $s = AS + YS$, $t = YT + BT$ and $u = BU + XU$. Since the lengths of the two tangents to a circle from the same point are equal, we have

\[
\begin{align*}
    r + t &= XR + AR + YT + BT \\
    &= XU + AS + YS + BU = u + s.
\end{align*}
\]

Furthermore, $\measuredangle A = 90^\circ = \measuredangle B$, so the Pythagorean theorem yields $r^2 + s^2 = (XY)^2 = t^2 + u^2$. Thus $r^2 - t^2 = u^2 - s^2$, and dividing through by $r + t = u + s$ yields $r - t = u - s$. Now adding this to the equation $r + t = u + s$, we obtain $2r = 2u$, so $r = u$, and hence $t = s$. Then $\triangle XAY \cong \triangle XBY$ by side-side-side, and thus line $XY$ bisects $\measuredangle AXB$. Since the bisector of the angle formed by two tangents of a circle from the same point goes through the center of the circle, it follows that $XY$ goes through the center, as required.

3. Let $a$ be a real number, and suppose that two of the three solutions of the cubic equation $x^3 + 3x^2 - 34x = a$ differ by 1. Find all possibilities for $a$.

**SOLUTION.** Suppose that both $r$ and $r + 1$ are solutions to the equation $x^3 + 3x^2 - 34x = a$. Then $r^3 + 3r^2 - 34r = a$, and also $(r + 1)^3 + 3(r + 1)^2 - 34(r + 1) = a$. Subtracting the first of these equations from the second yields $(3r^2 + 3r + 1) + 3(2r + 1) - 34 = 0$, and simplification yields $3r^2 + 9r - 30 = 0$. Thus $0 = r^2 + 3r - 10 = (r + 5)(r - 2)$. We conclude that either $r = -5$ or
4. Given a positive integer $n$, find all polynomials $f(x)$ such that $f(0) = 1$ and $f(x)^2 + 4x^{n+1} = g(x)^2 + 4x^n$ for some polynomial $g(x)$.

**SOLUTION.** Plugging $x = 0$ into the equation $f(x)^2 + 4x^{n+1} = g(x)^2 + 4x^n$, and recalling that $f(0) = 1$, we obtain $g(0)^2 = 1$. Replacing $g$ by $-g$ if necessary, we can assume that $g(0) = 1$. Now $(f + g)(f - g) = f^2 - g^2 = 4x^n - 4x^{n+1} = 4x^n(1 - x)$, and thus $f + g$ is a polynomial divisor of $4x^n(1 - x)$. But $f(x) + g(x)$ has the value 2 when $x = 0$, and thus $x$ cannot be a factor of $f + g$. It follows that either $f + g = c$ or $f + g = c(1 - x)$ for some constant $c$. Plugging in $x = 0$ yields $2 = c$, and we conclude that either $f + g = 2$ or $f + g = 2(1 - x)$. If $f + g = 2$, then since $(f + g)(f - g) = 4x^n(1 - x)$, we deduce that $f - g = 2x^n(1 - x)$. Adding these two equations, we get $2f = 2 + 2x^n(1 - x)$, and thus $f(x) = 1 + x^n - x^{n+1}$. The remaining possibility is that $f + g = 2(1 - x)$, and thus $f - g = 2x^n$. Adding, as before, we get $2f = 2(1 - x) + 2x^n$, and this time we have $f(x) = 1 - x + x^n$.

5. Given a positive integer $n$ other than 2, 3 or 5, show that a cardboard square can be cut into exactly $n$ smaller squares, not necessarily of the same size.

**SOLUTION.** Say that a positive integer $n$ is **good** if every square can be cut into $n$ squares, not necessarily of the same size. Clearly 1 is good and the following diagrams show that both 6 and 8 are good.

![Diagram of 6 squares](image1.png)

![Diagram of 8 squares](image2.png)

To show that every positive integer other than 2, 3 or 5 is good, suppose this is false, and let $n$ be the smallest integer other than 2, 3 or 5 that is not good. Since 1 is good, $n > 1$ and by assumption $n$ is not 2 or 3, so $n > 3$. We next observe that $n - 3$ is not 2, 3 or 5. Indeed, if $n - 3 = 2$ then $n = 5$, contrary to our assumption. On the other hand, if $n - 3 = 3$ or 5, then $n = 6$ or 8, and we know both of these are good.

Finally, we show that $n$ really is good after all. To this end, note that $n - 3$ is not 2, 3 or 5 and that it is smaller than $n$, so $n - 3$ must be good. Next, cut the original square into four equal smaller squares, and since $n - 3$ is good, cut one of those smaller squares into $n - 3$ squares. At this point, the original square has been cut into $n$ squares, so $n$ is good. We conclude from this contradiction that every positive integer different from 2, 3 or 5 is good.