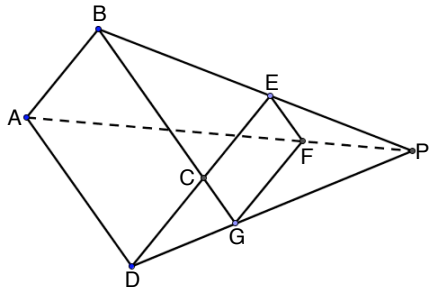
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

PROBLEM SET V (2008-2009) **FEBRUARY 2009**

1. Let us say that positive integers a and b are *related* if b^2+b+1 is a multiple of a and also a^2+a+1 is a multiple of b. (Thus, for example 1 and 1 are related, as are 1 and 3.) Show that there exists a pair of related integers both of which exceed 1000.

2. In the figure, $ABCD$ and $CEFG$ are parallelograms. Points B, C and G are collinear, as are points D, C and E, and lines $\overline{BE}$ and $\overline{DG}$ meet at point P. Show that points A, F and P are collinear.



3. Let a and b be positive integers, and suppose that $x<0$. Given that $(1+x^a+x^b)^2=3(1+x^{2a}+x^{2b})$, show that a and b are even, and find all possibilities for x.

4. The extraterrestrial aliens look just like humans, and although humans cannot distinguish aliens from humans, the aliens are able to recognize each other. Suppose that some aliens and humans are lined up, waiting to get into a theater. I, a human, have been told (by an alien) that more than half the people in line are aliens. I walk along the line, asking each person to estimate the number of aliens ahead of him in line. Of course, all of the aliens in the line give me correct answers, but the humans can only guess. Decide if it is possible for me to use the responses to pick out someone on the line that I can be sure is an alien.

5. Let $\square$ be an associative operation on the positive integers. In other words, if a and b are positive integers, then $a\square b$ is a positive integer, and $a\square(b\square c)=(a\square b)\square c$ for all positive integers a, b and c. Prove that there exist positive integers a and b such that $1\square a\square a\square b\square b\square b$ is **not** equal to $a+b$. (Note that because of the asociativity, expressions like $1\square a\square a\square b\square b\square b$ make sense without putting in any parentheses.)

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on this problem page. Remember that solutions usually require a proof or justification.

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Problem	**Score**
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