WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET V (2008-2009)

1. Let us say that positive integers a and b are *related* if $b^2 + b + 1$ is a multiple of a and also $a^2 + a + 1$ is a multiple of b. (Thus, for example 1 and 1 are related, as are 1 and 3.) Show that there exists a pair of related integers both of which exceed 1000.

SOLUTION. Suppose that a and b are related, where $a \le b$. (For example, we could have $a = 1$ and $b = 1$.) Suppose that p is a prime number that divides b. Then p divides $b^2 + b$, and thus p does not divide $b^2 + b + 1$. Since a does divide $b^2 + b + 1$, we see that p cannot divide a, and thus a and b have no common prime factor.

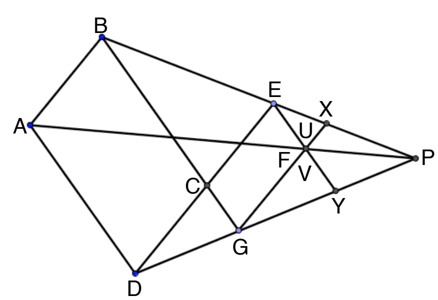
Now let $c = (b^2+b+1)/a$, so that c is an integer, and we have $b \le b(b/a) = b^2/a < c$. We argue now that b and c are related. Since $b^2 + b + 1 = ac$, it is clear that $b^2 + b + 1$ is a multiple of c, so we must show that c^2+c+1 is a multiple of b. Using the fact that a and b have no common prime factor, we see that it suffices to show that $a^2(c^2 + c + 1)$ is a multiple of b. Now $ac = b^2 + b + 1$, so we have

$$a^2(c^2 + c + 1) = (b^2 + b + 1)^2 + a(b^2 + b + 1) + a^2 = bN + (1 + a + a^2)\,,$$

for some integer N. Since $1+a+a^2$ is a multiple of b, it follows that $a^2(c^2+c+1)$ is also a multiple of b. Thus $c^2 + c + 1$ is a multiple of b, and therefore b and c are related.

Finally, consider the sequence $1, 1, 3, 13, 61, 291, 1393, 6673, \cdots$, constructed as follows. Each term c after the two initial 1s is computed from the previous two terms a and b by setting $c = (b^2 + b+1)/a$. Our argument shows that since 1 and 1 are related, so are 1 and 3, and since 1 and 3 are related, so are 3 and 13. Continuing like this, we see that 13 and 61 are related, that 61 and 291 are related, that 291 and 1393 are related, and that 1393 and 6673 are related. We have thus found two related numbers exceeding 1000.

2. In the figure, $ABCD$ and $CEFG$ are parallelograms. Points B, C and G are collinear, as are points D, C and E, and lines $\overline{BE}$ and $\overline{DG}$ meet at point P. Show that points A, F and P are collinear.



SOLUTION. Extend lines $\overline{GF}$ and $\overline{EF}$ to points X and Y, respectively, lying on $\overline{PB}$ and $\overline{PD}$, as shown. Let U be the point where $\overline{EY}$ meets $\overline{PA}$, and let V be the point where $\overline{GX}$ meets $\overline{PA}$. Our goal is to show that F lies on $\overline{PA}$, so it would be enough to show that $F = U$ or that $F = V$. In fact, we will show that $U = V$, so this point lies on both $\overline{EY}$ and $\overline{GX}$, and it will follow that $U = F = V$. Since U and V both lie on $\overline{PA}$, it is enough to show that $PU = PV$, and we proceed to establish this.

Since $\overline{EY}$ is parallel to $\overline{AD}$, we have $PU/PA = PY/PD$, and similarly, $PV/PA = PX/PB$. Also, since $\overline{EY}$ is parallel to $\overline{BG}$, we have $PE/PB = PY/PG$, and similarly, $PG/PD = PX/PE$. We now have

$$\frac{PU}{PA} = \frac{PY}{PD} = \frac{PY}{PG}\,\frac{PG}{PD} = \frac{PE}{PB}\,\frac{PX}{PE} = \frac{PX}{PB} = \frac{PV}{PA}\,.$$

It follows that $PU = PV$, as wanted.

3. Let a and b be positive integers, and suppose that $x < 0$. Given that $(1 + x^a + x^b)^2 = 3(1 + x^{2a} + x^{2b})$, show that a and b are even, and find all possibilities for x.

SOLUTION. Write $u = x^a$ and $v = x^b$. We have $(1 + u + v)^2 = 3(1 + u^2 + v^2)$, so if we expand the left side of this equation and then subtract 1, u^2 and v^2 from both sides, we get $2u + 2v + 2uv = 2 + 2u^2 + 2v^2$. Thus, we have

$$(u - v)^2 + (u - 1)^2 + (v - 1)^2 = u^2 - 2uv + v^2 + u^2 - 2u + 1 + v^2 - 2v + 1 = 0\,.$$

Since $(u - v)^2$, $(u - 1)^2$ and $(v - 1)^2$ are all nonnegative and their sum is 0, it follows that each of these quantities is 0, and thus $u = 1 = v$. Now $x < 0$ and $x^a = u = 1$, so a is even and $x = -1$. Similarly, b is even.

4. The extraterrestrial aliens look just like humans, and although humans cannot distinguish aliens from humans, the aliens are able to recognize each other. Suppose that some aliens and humans are lined up, waiting to get into a theater. I, a human, have been told (by an alien) that more than half the people in line are aliens. I walk along the line, asking each person to estimate the number of aliens ahead of him in line. Of course, all of the aliens in the line give me correct answers, but the humans can only guess. Decide if it is possible for me to use the responses to pick out someone on the line that I can be sure is an alien.

SOLUTION. Yes, I can pick out an alien; here is how. Let L be the number of individuals in line, and let n be the smallest integer such that $n > L/2$. Of course, I know L, so I can determine n. Now let a be the number of aliens in line, and note that although I do not know a, I do know that $a > L/2$, and thus $a \geq n$. Now in response to my question, the first alien in line says 0, the second alien says 1 and so on, until alien number a says $a - 1$. Among the L responses to my question, therefore, each of the numbers in the set $\{0, 1, 2, \ldots, a - 1\}$ occurs at least once. Since $n \leq a$, each of the n numbers in the set $\{0, 1, 2, \ldots, n - 1\}$ occurs at least once, and in fact, each of these numbers was the response of an alien. Since $2n > L$, it is not possible for every one of the numbers in the set $\{0, 1, 2, \ldots, n\}$ to occur twice as an answer, so at least one of these numbers was the response of just one individual in the line, and I know that that person must be an alien.

5. Let $\square$ be an associative operation on the positive integers. In other words, if a and b are positive integers, then $a \,\square\, b$ is a positive integer, and $a \,\square\, (b \,\square\, c) = (a \,\square\, b) \,\square\, c$ for all positive integers a, b and c. Prove that there exist positive integers a and b such that $1\square a\square a\square b\square b\square b$ is **not** equal to $a+b$. (Note that because of the asociativity, expressions like $1\square a\square a\square b\square b\square b$ make sense without putting in any parentheses.)

SOLUTION. Supposing that $1 \,\square\, a \,\square\, a \,\square\, b \,\square\, b \,\square\, b = a + b$ for all positive integers a and b, we work to derive a contradiction. To avoid writing messy formulas like $1 \,\square\, 1 \,\square\, 1 \,\square \cdots \square\, 1$, we will write $(1)^m$ to denote an expression of that form, where the number of 1s is exactly m. Now let $x = 1 \,\square\, 1 = (1)^2$. Then $x + 1 = 1 \,\square\, x \,\square\, x \,\square\, 1 \,\square\, 1 \,\square\, 1 = (1)^8$ and $1 + x = 1 \,\square\, 1 \,\square\, 1 \,\square\, x \,\square\, x \,\square\, x = (1)^9$, and we see that $(1)^8 = (1)^9$. Now consider $(1)^n$, where $n \geq 9$. We can replace the first nine 1s by eight 1s, and thus $(1)^n = (1)^{n-1}$. Repeating this argument a number of times, we deduce that $(1)^n = (1)^8$ for all integers $n \geq 8$, and in particular, $(1)^{20} = (1)^8$. Then

$$(1)^8 + 1 = 1 \,\square\, (1)^8 \,\square\, (1)^8 \,\square\, 1 \,\square\, 1 \,\square\, 1 = (1)^{20} = (1)^8\,,$$

and this is the desired contradiction.