WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2008-2009)

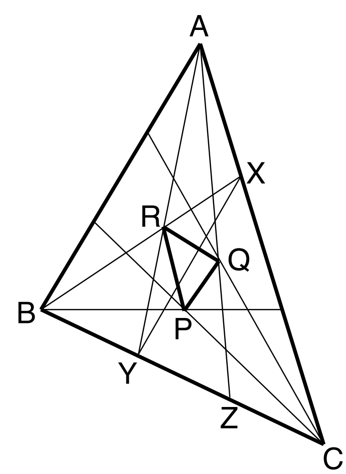
1. A string of 0s and 1s is said to be *palindromic* if it reads the same backwards and forwards. Thus, for example, 101 and 1001 are palindromic, but 1101 is not. Now let S be an infinitely long string of 0s and 1s with no beginning and no end. Show that S contains a palindromic string consisting of four or more consecutive symbols. Decide whether or not S must contain a palindromic string consisting of five or more consecutive symbols.

SOLUTION. If there are four consecutive 0s or 1s in S, we have a palindromic string, so we can assume that there are no four consecutive 0s or 1s in S. If S contains three consecutive 0s, they must therefore be preceded and followed by 1s, so we have the palindromic string 10001, and similarly, if S contains three consecutive 1s, it will contain the palindromic string 01110. We can assume, therefore, that S does not contain three consecutive 0s or 1s. If S contains two consecutive 0s, they must be preceded and followed by 1s, so S contains the palindromic string 1001, and similarly, if S contains two consecutive 1s, it contains the palindromic string 0110. We can thus assume that S does not contain two consecutive 0s or 1s, and thus the symbols 0 and 1 alternate in S, and S contains the palindromic string 01010. Thus in all cases, S contains a palindromic string of length at least four, indeed of length four or five.

There need not be a palindromic string of five or more consecutive symbols in S. To see this, suppose S is the string $\cdots 110010110010110010 \cdots$ formed by repeating the string 110010 infinitely often. Since the string 0011 does not occur in S, no palindromic string of consecutive symbols in S can contain 1100. All palindromic strings of consecutive symbols in S, therefore, must be part of 10010110. It easy to check that no five or more consecutive symbols in 10010110 are palindromic.

2. In the diagram, each side of $\triangle ABC$ is trisected, and each vertex is joined by line segments to the two trisection points of the opposite side. Let P, Q and R be the intersections of three pairs of these line segments, as shown. Prove that $\triangle PQR$ is similar to $\triangle ABC$, and compute its area as a fraction of the area of the big triangle.

SOLUTION. We will show that $RQ = (1/5)BC$. It will then follow in the same way that $QP = (1/5)AB$ and $PR = (1/5)CA$, and thus $\triangle PQR$ is similar to $\triangle ABC$ by side-side-side, and its area is $(1/5)^2 = 1/25$ of the area of $\triangle ABC$.



Label points X, Y and Z, as shown, and draw $\overline{XY}$. Since $AX = (1/3)AC$ and $BY = (1/3)BC$, it follows that $\overline{XY}$ is parallel to $\overline{AB}$ and $XY = (2/3)AB$. We see now that $\triangle RXY$ and $\triangle RBA$ are similar, and each side of $\triangle RXY$ has length 2/3 of the corresponding side of $\triangle RBA$. In particular, $RY = (2/3)RA$, and thus $AY = AR + RY = AR + (2/3)AR = (5/3)AR$. We now have $AR = (3/5)AY$, and similarly, $AQ = (3/5)AZ$. We conclude that $\triangle ARQ$ is similar to $\triangle AYZ$, and the length of each side of the smaller triangle is 3/5 the corresponding side of the larger triangle. In particular, $RQ = (3/5)YZ = (3/5)(1/3)BC = (1/5)BC$, as claimed.

3. Find all positive integers n such that $(n^3 + 100)/(n^2 + 100)$ is an integer.

SOLUTION. Clearly, $n = 1$ is a solution. We show that there is no other solution by deriving a contradiction from the assumption that n is a solution and $n > 1$.

By assumption, n^2+100 is a divisor of n^3+100, and of course, it is also a divisor of $n(n^2+100) = n^3 + 100n$. It follows that $n^2 + 100$ divides $(n^3 + 100n) - (n^3 + 100) = 100(n - 1)$. Now $n^2 + 100$ exceeds 100, so it cannot divide 100, and since it divides $100(n - 1)$, it follows that some divisor $d > 1$ of $n^2 + 100$ must divide $n - 1$. Then d divides $(n + 1)(n - 1) = n^2 - 1$, and since d also divides $n^2 + 100$, it follows that d divides $(n^2 + 100) - (n^2 - 1) = 101$. But 101 is prime and d is a divisor exceeding 1, so we must have $d = 101$.

We now know that 101 divides $n - 1$ and therefore $n \geq 102$, since $n > 1$. On the other hand, since $n^2 + 100$ divides $100(n - 1)$ and $n > 1$, it follows that $n^2 + 100 \leq 100(n - 1)$. Thus $n^2 \leq 100(n - 2) < 100n$, so $n < 100$ and we have the required contradiction. We conclude that $n = 1$ is the only solution.

4. Let $\square$ be an operation defined on the integers. (In other words, given integers a and b, we get an integer $a \square b$ determined by a and b.) Assume the following three axioms.

(1) $(a + b)(a \square b) = (a^2) \square (b^2)$ for all integers a and b.

(2) $(a \square b) + (b \square c) = a \square c$ for all integers a, b and c.

(3) $1 \square 0 = 1$.

Show that $a \square b = a - b$ for all integers a and b.

SOLUTION. By (2), we have $(0 \square 0) + (0 \square 0) = 0 \square 0$, and thus $0 \square 0 = 0$. Using (2) again, we have $0 = 0 \square 0 = (0 \square a) + (a \square 0)$, and thus $0 \square a = -(a \square 0)$. Now let a and b be arbitrary integers. Then (2) yields $(a \square 0) + (0 \square b) = a \square b$, and if we multiply both sides by $a + b$, we obtain

$$a(a \square 0) + b(a \square 0) + a(0 \square b) + b(0 \square b) = (a + b)(a \square b) = (a^2) \square (b^2) \qquad (*)$$

by axiom (1). Furthermore, by (1) again, $a(a \square 0) = (a + 0)(a \square 0) = (a^2) \square 0$, and $b(0 \square b) = (0 + b)(0 \square b) = 0 \square (b^2)$. The sum of the first and fourth terms on the left of $(*)$ is therefore $((a^2) \square 0) + (0 \square (b^2)) = (a^2) \square (b^2)$, which is the same as the right side of $(*)$. It follows that $b(a \square 0) + a(0 \square b) = 0$, and thus since $(b \square 0) = -(0 \square b)$, we have $b(a \square 0) = a(b \square 0)$ for all a and b.

Finally, letting $b = 1$, we have $a \square 0 = a(1 \square 0) = a$, using (3). This holds for all integers a, so we can write $a \square b = (a \square 0) + (0 \square b) = (a \square 0) - (b \square 0) = a - b$, as required.

5. Let n be a positive integer. Show that $K_n = 2^{2n-1} - 9n^2 + 21n - 14$ is a multiple of 27.

SOLUTION. Write $D_n = K_{n+1} - 4{\cdot}K_n$ for $n \geq 1$. Suppose we can prove that all of the numbers D_n are multiples of 27. Since $K_{n+1} = 4{\cdot}K_n + D_n$, it follows that if K_n is a multiple of 27, then so is K_{n+1}. Since $K_1 = 2 - 9 + 21 - 14 = 0$ is indeed a multiple of 27, it follows that K_2 is also, and thus so is K_3 a multiple of 27. Continuing in this way, it follows that all K_n are multiples of 27.

It remains to understand D_n. For this, observe that

$$\begin{aligned} D_n &= K_{n+1} - 4{\cdot}K_n \\ &= (2^{2n+1} - 4{\cdot}2^{2n-1}) - 9((n + 1)^2 - 4{\cdot}n^2) + 21((n + 1) - 4{\cdot}n) - 14(1 - 4{\cdot}1) \\ &= 0 - 9(-3n^2 + 2n + 1) + 21(1 - 3n) - 14(-3) \\ &= 27(n^2 - 3n + 2), \end{aligned}$$

and therefore D_n is indeed a multiple of 27, as wanted.