WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2008-2009)

1. Let x, y and z be positive real numbers. Show that

$$(xy + yz + zx)(x + y + z) \geq 9xyz\,.$$

SOLUTION. First, we observe that if x and y are any two positive real numbers, then $x/y+y/x \geq 2$. To see this, start with the inequality $(x - y)^2 \geq 0$, which yields $x^2 + y^2 \geq 2xy$. Now dividing by the positive number xy, we get $x/y + y/x \geq 2$, as wanted.

Applying this fact three times, we get

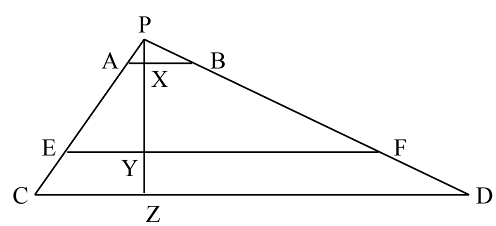
$$\frac{x}{y} + \frac{y}{x} + \frac{x}{z} + \frac{z}{x} + \frac{y}{z} + \frac{z}{y} \geq 6\,,$$

and multiplication by the positive number xyz yields

$$x^2z + y^2z + x^2y + z^2y + y^2x + z^2x \geq 6xyz\,.$$

If we add $3xyz$ to both sides, the left side factors to yield $(xy + yz + zx)(x + y + z) \geq 9xyz$.

2. The parallel sides of a trapezoid have lengths 1 and 7, and the area of the trapezoid is divided into two equal parts by a line segment parallel to those two sides. Find the length of that line segment, and prove that your answer is correct.



SOLUTION. Let $ABDC$ be a trapezoid with parallel sides $\overline{AB}$ and $\overline{CD}$, and let $\overline{EF}$ be parallel to $\overline{AB}$ and $\overline{CD}$ and cut the area of the trapezoid in half. Let $u = AB$, $v = CD$ and $t = EF$. We show below that $2t^2 = u^2 + v^2$, so if $u = 1$ and $v = 7$, we get $2t^2 = 50$, and thus $t = 5$.

We can assume that $AB < CD$, and we let P be the point where $\overline{AC}$ meets $\overline{BD}$, as shown. Draw a line through P perpendicular to $\overline{AB}$, $\overline{EF}$ and $\overline{CD}$, meeting those parallel lines at points X, Y and Z, respectively, and let $x = PX$, $y = PY$ and $z = PZ$.

Using similar triangles, we see that $x/u = y/t = z/v$. Writing r to denote the common value of these three ratios, we get $x = ur$, $y = tr$ and $z = vr$. Since the areas of trapezoids $ABFE$ and $EFDC$ are equal, we have $(t + u)(y - x) = (v + t)(z - y)$. (Each side of this equation equals twice the area of one of these trapezoids.) Substitution of ur, tr and vr for x, y and z yields $(t + u)(tr - ur) = (v + t)(vr - tr)$. Now we divide by r to get $t^2 - u^2 = v^2 - t^2$, and thus $u^2 + v^2 = 2t^2$, as wanted.

3. Each member of a sports team is a friend of at least six others on the team, and yet there are no four members of the team such that each of them is a friend of the other three. Find (with proof, of course) the smallest size team for which this is possible. (You should assume that the "friend" relationship is symmetric. In other words, if A is a friend of B, then also, B is a friend of A.)

SOLUTION. The smallest possible team size is 9. We first show that with nine team members, it is possible to have four of them who are not all friends of each other. Suppose the team members are divided into three sets of three, say $\{A, B, C\}$, $\{D, E, F\}$ and $\{G, H, I\}$. Assume that each person is a friend of the six people in the two sets of which he is not a member, but that he is not a friend of anyone in his own set. Then every member has six friends, but no matter how we choose four people, there must be at least two of them in the same set of three. Those two team members are not friends, and thus no four people on the team are all friends of each other, as required.

Now we show that for a team with eight members, in which each person has at least six friends, there must exist a set of four mutual friends. For each team member X, there is at most one member other than X himself who is not a friend of X. Now choose any team member A, and if someone (other than A) is not a friend of A, send that person back to the locker room. This leaves at least seven people, including A. Now choose a second person B, and dismiss the (at most one) person other than B himself who is not a friend of B. There are at least six survivors including A and B, and we select a third person C from among them, and we send away the (at most one) person other than C who is not C's friend. This leaves at least five survivors, including A, B, and C, and we choose a fourth person D from among them. Now every two people in the set $\{A, B, C, D\}$ are friends.

4. Note that $654/545 = 6/5$, so that one can "cancel" the 54 in the numerator and denominator of $654/545$ without changing the value of the fraction. Now consider the fraction $6545454\cdots54/545454\cdots545$, where there are n copies of 54 following the digit 6 of the numerator, and the same number n of copies of 54 preceding the final digit 5 of the denominator. Prove that this fraction is equal to $6/5$ for every positive integer n.

SOLUTION. Let $a = 65454\cdots54$, where there are n copies of "54". Observe that a is divisible by 6, and the quotient $a/6$ is equal to $10909\cdots09$, where there are n copies of "09". Now let $b = 5454\cdots545$, where there are n copies of "54" before the final 5. We see that $b/5 = 10909\cdots09$, where again, there are n copies of "09". Thus $a = 6m$ and $b = 5m$, where $m = 10909\cdots09$, and thus $a/b = (6m)/(5m) = 6/5$, as wanted.

5. Recall that $n!$ (pronounced "n-factorial") is defined to be the product of all of the integers from 1 through n. Prove that $n! \le 2(n/2)^n$ for all positive integers n.

SOLUTION. We use induction on n. First observe that the desired inequality holds when $n = 1$ since in that case $n! = 1$ and $2(n/2)^n = 1$. Next, we show below that if the desired inequality holds for some number n, then it also holds for the next number, $n+1$. Since it holds for $n = 1$, it follows that it holds for $n = 2$, and thus it holds for $n = 3$ and so on, and hence it holds for all n.

Note that if a is positive, then $(1+a)^n = 1+na+\cdots$, where all of the omitted terms are positive, and thus $(1+a)^n \ge 1+na$. In particular, taking $a = 1/n$, we get $((n+1)/n)^n = (1+1/n)^n \ge 2$.

Suppose now that the desired inequality is known for some value of n, so that we have $n! \le 2(n/2)^n$. Since $2 \le ((n+1)/n)^n$, this yields

$$n! \le 2\Big(\frac{n}{2}\Big)^n \le \Big(\frac{n+1}{n}\Big)^n \Big(\frac{n}{2}\Big)^n = \Big(\frac{n+1}{2}\Big)^n.$$

Now multiply both sides by $n+1$ to get

$$(n+1)! \le \frac{(n+1)^{n+1}}{2^n} = 2\Big(\frac{n+1}{2}\Big)^{n+1}.$$

This is exactly the desired inequality for $n+1$.