WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET V (2007-2008)

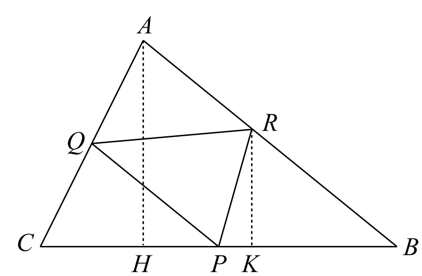
1. Fix a real number $k > 0$. Let $x_1, x_2, \ldots, x_n$ be n positive real numbers and suppose that $x_1(2k - x_2) = k^2$, $x_2(2k - x_3) = k^2$, $x_3(2k - x_4) = k^2, \ldots$, and $x_n(2k - x_1) = k^2$. Show that all x_i are equal to k.

SOLUTION. Let $x_1, x_2, \ldots, x_n$ be n positive real numbers that satisfy the given n equations. Since each $x_i > 0$ and $k^2 > 0$, it follows that $2k - x_i > 0$ for all i. Next, notice that for any real number x with $x > 0$ and $2k - x > 0$, we have $0 < x(2k - x) = k^2 - (k - x)^2 \le k^2$, with equality at the right if and only if $x = k$. Finally, if we multiply the n given equations and reorder the factors, we obtain

$$(k^2)^n = \prod_{i=1}^{n} x_i(2k - x_i) \le (k^2)^n,$$

where the right-hand inequality follows from $0 < x_i(2k - x_i) \le k^2$ for all i. Certainly, the right-hand inequality must be equality, so $x_i(2k - x_i) = k^2$, and hence $x_i = k$ for all i.

2. Given triangle $\triangle ABC$, let point P lie on side $\overline{BC}$, point Q on side $\overline{CA}$ and point R on side $\overline{AB}$. Suppose that the four triangles $\triangle ARQ$, $\triangle BPR$, $\triangle CQP$ and $\triangle PQR$ all have equal area. Show that P, Q and R are the midpoints of the respective sides.



SOLUTION. Since all four small triangles have the same area, each of these areas is precisely one quarter of the area of $\triangle ABC$. Let us write $x = RB/AB$, $y = PC/BC$ and $z = QA/CA$, so that x, y and z are positive real numbers. The goal is to show that each of these numbers is equal to $1/2$. To this end, we compare the area of $\triangle BPR$ to that of $\triangle ABC$. First, the ratio of their bases BP to BC is equal to $(BC - PC)/BC = 1 - y$. Next, if $\overline{AH}$ and $\overline{RK}$ are the altitudes as indicated in the diagram, then certainly $\triangle RKB$ is similar to $\triangle AHB$, and hence $RK/AH = RB/AB = x$. Thus

$$x(1 - y) = \frac{(1/2)\cdot RK\cdot BP}{(1/2)\cdot AH\cdot BC} = \frac{\text{area } \triangle BPR}{\text{area } \triangle ABC} = 1/4.$$

Similarly, we get $y(1 - z) = 1/4$ and $z(1 - x) = 1/4$. The result of the previous problem with $k = 1/2$, $n = 3$, and with the unknowns x, y and z, now implies that $x = y = z = k = 1/2$. Consequently, each of P, Q and R is the midpoint of its respective side of $\triangle ABC$.

3. Let d and n be positive integers, and let r be coprime to d. Show that for some integer k, the number $r + kd$ is coprime to n.

SOLUTION. Let k be the product of all the prime divisors of n that do not divide r. If there are no such primes, then we take $k = 1$. We show that $r + kd$ and n are relatively prime. If this is not the case, then there exists a prime p that divides both $r + kd$ and n. Now if p does not divide r, then since p divides n, the definition of k implies that p divides k. But then p divides $r + kd$ and kd, so p divides $(r + kd) - kd = r$, a contradiction. On the other hand, if p divides r, then since p divides n, the definition of k implies that p does not divide k. Furthermore, p divides $r + kd$

and r, so p divides $(r + kd) - r = kd$ and, since p does not divide k, we conclude that p divides d. But then p divides both d and r, and this contradicts our assumption that d and r are relatively prime. We have shown that p cannot exist and hence $r + kd$ and n are coprime, as required.

The conclusion of this problem is also a consequence of a major theorem in number theory. Specifically, Dirichlet's Theorem asserts that if d and r are coprime, then there are infinitely many positive integers k such that $r + kd$ is a prime number. In particular, for some k, $r + kd$ is a prime number different from the primes dividing n, and hence it is relatively prime to n.

4. We call the squares of a 5 by 5 checkerboard adjacent if they share a side. What is the minimum number of squares that one can color yellow on such a 5 by 5 board so that every non-yellow square is adjacent to a yellow square?

SOLUTION. The answer is 7. We first show that no pattern of 6 yellow squares will suffice, and then we offer a pattern of 7 yellow square that does suffice.

Suppose, by way of contradiction, that precisely six squares of the 5×5 board are colored yellow and that every non-yellow square is adjacent to a yellow one. If some column contains no yellow squares, then each square is non-yellow and consequently a square to its left or right must be yellow. This shows that each row contains a yellow square and hence, by row-column symmetry, we can now assume that each column contains at least one yellow square. Since, each of the five columns contains a yellow square and since there are only six yellow squares in all, it follows that four of the columns have precisely one yellow square, while the remaining column has two. By right-left symmetry, we can assume that the column with two yellow squares is number 3, 4 or 5 counting from the left. In particular, columns 1 and 2 each contain just one yellow square. Hence column 1 has four non-yellow squares. Now at most one of these can be adjacent to the yellow square in column 2, and at most two of these can be adjacent to the yellow square in column 1. Thus one of the non-yellow squares in column 1 is not adjacent to any yellow square, and this is the required contradiction. We conclude that six yellow squares are not sufficient, and it is easy to see that the diagram on the right has a pattern of seven yellow squares that does suffice.

		Y		
Y				Y
		Y		
Y				Y
		Y		

5. Do there exist polynomials $P(x)$, $Q(x)$ and $R(x)$ having integer coefficients such that $P(x)$ is the product $P(x) = Q(x)R(x)$, all coefficients of $P(x)$ are 0, 1 or -1, and one of the coefficients of $Q(x)$ is 2008 ?

SOLUTION. There are many examples of such polynomials. We take $R(x) = x - 1$ and let $Q(x)$ be an "up and down" polynomial having first increasing coefficients starting at 1 and then decreasing coefficients ending at 1. Of course, each increase or decrease is equal to 1. Specifically, choose any integer $a \geq 2007$ and let $Q(x)$ be the polynomial of degree $2a$ given by

$$\begin{aligned} Q(x) = 1 + 2x + 3x^2 + \cdots + &(a-1)x^{a-2} + ax^{a-1} + (a+1)x^a \\ &+ ax^{a+1} + (a-1)x^{a+2} + \cdots + 3x^{2a-2} + 2x^{2a-1} + 1x^{2a}. \end{aligned}$$

Then $P(x) = Q(x)R(x) = Q(x)(x-1)$ has degree $2a+1$ with leading coefficient 1 and constant term -1. Furthermore, the remaining coefficients, those corresponding to x^i with $1 \leq i \leq 2a$, are clearly each equal to a difference of two adjacent coefficients of $Q(x)$, and hence are equal to ± 1, as required. Since $a + 1 \geq 2008$, we note that 2008 occurs as a coefficient of $Q(x)$.