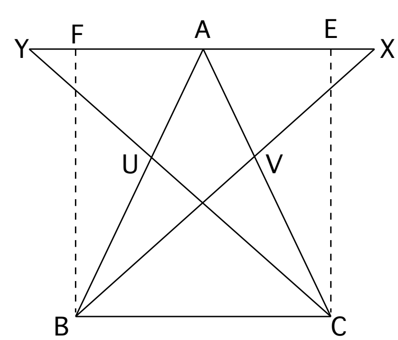
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET III (2007-2008)

1. Numerican University offers courses in 5 subjects, call them A, B, C, D, and E. Every student is interested in exactly one of these, but some subjects require taking courses in certain other subjects simultaneously. No student takes courses in subjects he is not interested in unless he is forced to do so. Currently, there are 200 students at NU and the enrollment in courses A, B, C, D, and E is 100, 110, 120, 130, and 150, respectively. If a student is interested in A, how many other courses must he take? What about B?

SOLUTION. We first consider A and note that this course has the lowest enrollment of all. If some other course, say X, required that one also take A, then everyone enrolled in X would also be enrolled in A, and this cannot occur since the enrollment in A is strictly smaller than that of X. Now suppose Y is any course other than A and note that the enrollment in A plus the enrollment in Y is larger than 200, the total number of students at NU. Thus there must be a student taking both of these courses. If this particular student really wanted to take course Z, then Z must force him to take both A and Y. Thus, by the above, Z is equal to A, and then A forces him to take Y. We conclude that any student who wants to take A must take all four of the remaining courses.

It follows that precisely 100 students want to take A, and these students must also take B, C, D, and E. If we eliminate these 100 students from consideration, then we are left with 100 students at NU whose enrollments in B, C, D, and E are 10, 20, 30, and 50, respectively. Since each student is interested in exactly one course, it is clear that the total enrollment here is equal to the number of students plus the forced enrollment. Furthermore, in our situation, the sum of the course enrollments is 110, just 10 more than the number of students we are considering. If one of the courses C, D, or E required a second course, then the forced enrollment would be at least 20 and the total enrollment would be at least 120, which is not the case. Thus C, D, and E do not require additional courses. Finally, if B requires k additional courses, then the total forced enrollment is $10k$, so the total enrollment is $100 + 10k = 110$, and therefore $k = 1$. In other words, any student who wants to take B must also take one additional course.

2. In the figure, $\triangle ABC$ is given, and line $\overline{XY}$ is drawn through A, parallel to $\overline{BC}$. Lines $\overline{XB}$ and $\overline{YC}$ are drawn,meeting the sides of the given triangle at V and U, respectively. If $AX = AY$ and $BV = CU$, prove that $AB = AC$.



SOLUTION. Since $\overline{AY}$ is parallel to $\overline{BC}$, we see that $\angle AYU = \angle BCU$ and $\angle YAU = \angle CBU$. It follows that $\triangle AYU$ is similar to $\triangle BCU$, and thus $AY/BC = YU/CU$. Similarly, working with $\triangle AXV$ and $\triangle CBV$, we see that $AX/CB = XV/BV$. But we are given that $AX = AY$ and $BV = CU$, and thus

$$YU = \frac{(CU)(AY)}{BC} = \frac{(BV)(AX)}{CB} = XV\,,$$

and it follows that $CY = CU + UY = BV + VX = BX$.

Now as shown, drop perpendiculars $\overline{BF}$ and $\overline{CE}$ from B and C to $\overline{XY}$ (which may have to be extended, depending on how the diagram is drawn). Since $\overline{XY}$ and $\overline{BC}$ are parallel, it follows that $BF = CE$. We have shown that $CY = BX$, and thus the right triangles $\triangle XFB$ and $\triangle YEC$ are congruent by the hypotenuse-side congruence criterion, and we deduce that $\angle X = \angle Y$. Since we were given that $XA = YA$ and we showed that $XV = YU$, it follows that $\triangle XAV \cong \triangle YAU$ by side-angle-side, and thus $\angle YAU = \angle XAV$. We observed previously that $\angle YAU = \angle CBU$, and similarly, $\angle XAV = \angle BCV$, and thus $\angle CBU = \angle BCV$. These, however, are the base angles of the original $\triangle ABC$, and since they are equal, $AB = AC$, as wanted.

3. Find all positive integers k such that the number $1444\ldots44$, having exactly k digits equal to 4, is a perfect square.

SOLUTION. Suppose first that $k \geq 4$. If $n = 1444\ldots44$ is a perfect square, then it must be the square of an even integer, say $2m$. Canceling a factor of 4 from the equation $n = (2m)^2 = 4m^2$, we obtain $361\ldots11 = m^2$. Here the number $361\ldots11$ has $k-2 \geq 2$ digits equal to 1. Now we see that m is odd, so $m = 2a+1$ and $m^2 = 4a^2+4a+1$ leaves a remainder of 1 when divided by 4. On the other hand, the remainder when $361\ldots11$ is divided by 4, is the same as when 11 is divided by 4, and this remainder is 3. Thus $361\ldots11$ cannot be a perfect square.

It follows that $k \leq 3$. Since 14 is not a perfect square, but $144 = 12^2$ and $1444 = 38^2$, we see that $k = 2$ and $k = 3$ are the only possibilities.

4. Find all real numbers x that satisfy the equation

$$x = |x - |x - |x - \cdots |x - |x-1|| \cdots |||$$

where there are 100 absolute values on the right hand side.

SOLUTION. Since $|a| = \pm a$ for any real number a, the right hand side of the above equation looks like $\pm(x - \pm(x - \cdots \pm (x - \pm(x-1)) \cdots)))$, where not all combinations of the $\pm$ signs are possible. It follows that the right hand side looks like $kx \pm 1$ where k is an even integer, since 100 is even, and $-100 \leq k \leq 100$. Thus the equation becomes $x = kx \pm 1$, so $x = \pm(1/n)$ where n is an odd integer between 1 and 101. Since x must clearly be nonnegative, we have $x = 1/n$ for some $n = 1, 3, \ldots, 101$.

Next, we show that these numbers x do indeed satisfy the equation. To start with, if $x = 1/n$, then $|x-1| = 1-x = 1-1/n$, so $|x-|x-1|| = |1/n-(1-1/n)| = 1-2/n$, and $|x-|x-|x-1||| = |1/n-(1-2/n)| = 1-3/n$. Continuing in this manner, it follows that after n iterations we obtain $1 - n/n = 0$. The remaining iterations now clearly alternate between $1/n$ and 0. Since n is odd, there are an odd number of remaining iterations, so that at the end we get $1/n = x$.

5. Let $x_1, x_2, \ldots, x_n$ be n nonnegative real numbers that sum to 1. If $n \geq 4$, find the largest possible value for the expression

$$y = x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

SOLUTION. If $x_1 = x_2 = 1/2$ and the remaining x_i's are 0, then $y = 1/4$, and we show below that this is the largest possible value for y. To this end, first note that if $u+v = 1$, then $4uv = (u+v)^2 - (u-v)^2 = 1-(u-v)^2 \leq 1$ and thus $uv \leq 1/4$.

Now if $n \geq 4$ is even, set $u = x_1+x_3+\cdots+x_{n-1}$ and $v = x_2+x_4+\cdots+x_n$. Then $u+v = 1$ and uv contains all the x_ix_j products in y plus some additional products that are nonnegative since each x_i is nonnegative. Thus $y \leq uv \leq 1/4$. Finally, if n is odd, then, by symmetry, we can suppose that x_1 is the smallest of the x_i's. In this case, if $u = x_1+x_3+\cdots+x_n$ and $v = x_2+x_4+\cdots+x_{n-1}$, then $u+v = 1$ and uv contains all x_ix_j products in y other than x_nx_1. On the other hand, uv also contains the product x_nx_2, and this product does not show up in y, since $n > 3$. Furthermore, $x_nx_1 \leq x_nx_2$ since $x_1 \leq x_2$. Thus again we have $y \leq uv \leq 1/4$, as required.