WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2007-2008)

1. Suppose that $a \,\square\, b$ is an integer for each integer a and b. Assume that, for all integers a and b, we have $(a+1) \,\square\, b - (a-1) \,\square\, b = 4a$ and $b \,\square\, a = -(a \,\square\, b)$. If $1 \,\square\, 0 = 1$, find the value of $101 \,\square\, 100$.

SOLUTION. We will use two well-known facts that are special cases of the formula for the sum of the numbers in an arithmetic progression. Specifically, if n is a positive integer then $2+4+6+\cdots+2n = n(n+1)$ and $1+3+5+\cdots+(2n-1) = n^2$.

Now fix b and note that $101 \,\square\, b - 99 \,\square\, b = 4{\cdot}100$, $99 \,\square\, b - 97 \,\square\, b = 4{\cdot}98$, $97 \,\square\, b - 95 \,\square\, b = 4{\cdot}96$, and so on until we reach $3 \,\square\, b - 1 \,\square\, b = 4{\cdot}2$. When we add these equations, all the middle terms cancel in pairs, so we obtain

$$101 \,\square\, b - 1 \,\square\, b = 4(2+4+6+\cdots+100) = 4(50)(51).$$

Similarly, $100 \,\square\, b - 98 \,\square\, b = 4{\cdot}99$, $98 \,\square\, b - 96 \,\square\, b = 4{\cdot}97$, $96 \,\square\, b - 94 \,\square\, b = 4{\cdot}95$, and so on until we reach $2 \,\square\, b - 0 \,\square\, b = 4{\cdot}1$. This time, when we add these equations, we obtain

$$100 \,\square\, b - 0 \,\square\, b = 4(1+3+5+\cdots+99) = 4(50)^2.$$

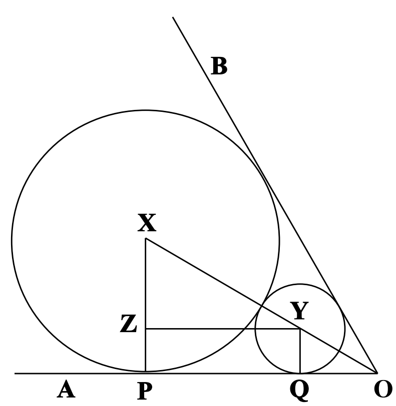
In particular, when $b=1$, the latter yields $100 \,\square\, 1 = 4(50)^2 + 0 \,\square\, 1 = 4(50)^2 - 1 \,\square\, 0 = 4(50)^2 - 1$.

Finally, with $b = 100$, we have

$$101 \,\square\, 100 = 4(50)(51) + 1 \,\square\, 100 = 4(50)(51) - 100 \,\square\, 1 = 4(50)(51) - 4(50)^2 + 1 = 201.$$

Using the same argument, it can be shown that $a \,\square\, b = a^2 - b^2$ for all integers a and b.

2. Three circles are tangent to the sides of the 60° angle $\angle AOB$ and also to each other, as indicated in the diagram. If the radius of the smallest circle is 1, determine the radius of the largest circle.



SOLUTION. Consider the two circle situation as indicated in the diagram, where the larger circle has radius R and the smaller has radius r. Note that the centers X and Y of the circles are both on the angle bisector of $\angle AOB$. Hence $\overline{XYO}$ is a straight line and $\angle XOA = 30^\circ$. Since $\overline{XY}$ goes through the point of tangency of the two circles, its length is $XY = R + r$. Now. let P and Q denote the points of tangency of the larger and smaller circle with line $\overline{AO}$. Then $XP = R$, $YQ = r$, and both these radius lines are perpendicular to $\overline{AO}$. Finally, let Z the the point on $\overline{XP}$ where $\overline{ZY}$ is parallel to $\overline{AO}$. Then $\overline{ZY}$ is perpendicular to $\overline{XP}$ and has $\overline{XZ}$ has length $XZ = R - r$. Since $\angle XYZ = \angle XOA = 30^\circ$, we see that $R + r = XY = 2(XZ) = 2(R+r)$, and hence $R = 3r$. Thus, the radius of the larger circle is 3 times the radius of the smaller circle.

Finally, in the original problem, we have three circles, with the smallest having radius 1. Thus the above implies that the middle circle has radius $3{\cdot}1 = 3$. Furthermore, by comparing the middle circle to the largest one, we conclude from the above that the radius of the largest circle is 3 times that of the middle circle. Thus the radius of the largest circle is $3{\cdot}3 = 9$.

3. Find all positive integers n such that the product $n(n + 16)$ is a perfect square.

SOLUTION. Observe that $n^2 < n(n+16) < (n+8)^2$. Therefore, if $n(n+8)$ is a perfect square, it must be the square of an integer of the form $n + k$ with $1 \le k \le 7$. Now, $n(n + 16) = (n + k)^2$ yields $16n = 2nk + k^2$, so k is even and hence $k = 2\ell$ with $\ell = 1$, 2 or 3. Substituting 2ℓ for k and dividing by 4, we obtain $n(4 - \ell) = \ell^2$. Now $\ell = 1$ gives $n = 1/3$, and this is not allowed. On the other hand, $\ell = 2$ yields $n = 2$, while $\ell = 3$ gives us $n = 9$. Note that, when $n = 2$, we have $n(n + 16) = 2{\cdot}18 = 6^2$, and when $n = 9$, we have $n(n + 16) = 9{\cdot}25 = 15^2$. Thus $n = 2$ and $n = 9$ are the unique solutions.

4. Let x, y and z be real numbers with $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and $x + y + z = 2$. Prove that

$$2 \ge \frac{x}{x+1} + \frac{y}{y+1} + \frac{z}{z+1} \ge 1\,.$$

SOLUTION. Since $0 \le x \le 1$, we have $1 \le x + 1 \le 2$ and hence $1 \ge 1/(x + 1) \ge 1/2$. Thus, since $x \ge 0$, we have $x \ge x/(x+1) \ge x/2$. Similarly $y \ge y/(y+1) \ge y/2$ and $z \ge z/(z+1) \ge z/2$. By adding these three inequalities and using the fact that $x + y + z = 2$, we obtain

$$2 = x + y + z \ge \frac{x}{x+1} + \frac{y}{y+1} + \frac{z}{z+1} \ge \frac{1}{2}(x + y + z) = 1\,,$$

as required.

5. In the country of Numerica there are 10 large cities some of which are connected by highways. Each highway goes from one city to another and has no intermediate exits. Numericans never fly from one city to another if they can drive there, even if this involves traveling through several different cities. At the beginning of each year, Numerican Airlines (NA) can establish at no cost any number of direct flight routes joining two cities and it can finance road construction on any number of existing highways, thereby making those highways impassable. The profit from any flight route that anybody bothers to use is 1 million per year. The cost of financing a road construction on a single highway varies from year to year. Last year it was 5 million and NA could not make a profit despite its best efforts. This year the cost is down to 4 million. Prove that NA's profit for this year will not be greater than 9 million.

SOLUTION. Suppose, by way of contradiction, that NA's profit is at least 10 million this year. Then NA could have used the same set of routes and road constructions last year. With this scheme, it would have earned the same amount of money from each route, but it would have paid 1 million more for each road construction project. Since we know that it was impossible for NA to make a profit last year no matter which scheme it used, this scheme would not have yielded a profit. Why not? It must be because the scheme contains at least 10 road constructions which would have added 10 million to NA's cost.

This year, with at least 10 road constructions, NA's cost is at least $4{\cdot}10 = 40$ million. Thus, in order for NA to have a profit of 10 or more million, it must earn at least 50 million from its routes. In other words, there must be at least 50 routes joining the 10 cities. But the maximum number of possible routes is the number of ways of choosing 2 out of the 10 cities, and this is given by $(10{\cdot}9)/2 = 45$, a contraction. We conclude that NA's profit this year can be at most 9 million.