WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2006-2007)

1. Let x, y and z be positive numbers such that $x + y + z \leq 1$. Show that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9$$

SOLUTION. First, we argue that if x and y are positive, then $x/y + y/x \geq 2$. To see this, observe that

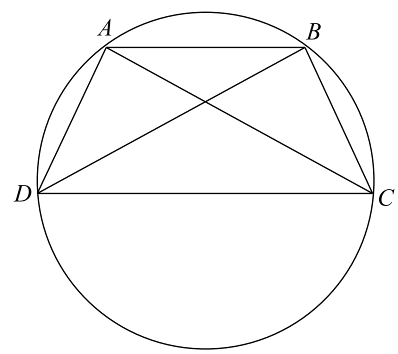
$$\frac{x}{y} - 2 + \frac{y}{x} = \frac{x^2 - 2xy + y^2}{xy} = \frac{(x-y)^2}{xy}$$

and this is nonnegative since xy is positive and $(x-y)^2$ is nonnegative. Now let x, y and z be positive. Then

$$(x+y+z)\Big(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\Big) = 3 + \Big(\frac{x}{y} + \frac{y}{x}\Big) + \Big(\frac{x}{z} + \frac{z}{x}\Big) + \Big(\frac{y}{z} + \frac{z}{y}\Big).$$

Each of the three quantities in parentheses on the right is at least 2 by the previous remark, and so the whole right side is at least $3 + 2 + 2 + 2 = 9$. Assuming that $x + y + z \leq 1$, therefore, we see that the second factor on the left is at least 9, as wanted.

2. Let $ABCD$ be a trapezoid, with $\overline{AB}$ parallel to $\overline{CD}$. Draw diagonals $\overline{AC}$ and $\overline{BD}$ and assume that $\angle DAC = \angle DBC$. Show that $AC = BD$.



SOLUTION. First, draw the circle through points A, C and D. Since $\angle DAC = \angle DBC$, it follows that point B must also lie on this circle. Because $\overline{AB}$ and $\overline{CD}$ are parallel, we see that $\angle ABD = \angle BDC$, and thus the arcs they subtend, which are $\overset{\frown}{AD}$ and $\overset{\frown}{BC}$, must be equal. Hence the corresponding chords $\overline{AD}$ and $\overline{BC}$ have equal lengths. Also, $\angle ADB = \angle ACB$ since these angles subtend the same arc, namely $\overset{\frown}{AB}$. It follows that $\triangle ADC \cong \triangle BCD$ by angle-side-angle, and we conclude that $AC = BD$, as required.

3. Suppose that every point in the plane is colored either red or blue. Prove that there exists an equilateral triangle whose three vertices all have the same color.

SOLUTION. First, choose two points A and B that have the same color, which we can assume is red. We can choose a coordinate system (which we use to describe other points) by taking the x-axis to be the line through A and B, with A at the origin and B at $(2, 0)$. Now let C be the point $(1, \sqrt{3})$ and note that A, B and C form an equilateral triangle. Since A and B are red, we see that if C is also red, we are done. Hence we can assume that C is blue. Similarly, we can assume that point D at $(1, -\sqrt{3})$ is blue.

Now let E be at $(4, 0)$ and note that $\triangle CDE$ is equilateral since each side has length $2\sqrt{3}$. Since C and D are blue, we can assume that E is red. Now let F be at $(3, \sqrt{3})$, so that $\triangle BEF$ is equilateral. Since B and E are red, we can assume that F is blue. Finally, let G be at $(2, 2\sqrt{3})$ and observe that $\triangle CFG$ and $\triangle AEG$ are both equilateral. If G is red, then $\triangle AEG$ is the desired equilateral triangle, and if G is blue, then $\triangle CFG$ is the triangle we seek.

4. Let us say that an infinite set S of positive integers is *anticlosed* if whenever x and y are two different members of S, their sum $x+y$ is not a member of S. Prove that the set of all positive integers is the union of some infinite collection of anticlosed subsets. Decide whether or not the set of all positive integers is the union of two anticlosed subsets.

SOLUTION. Given an integer $e \geq 0$, let S_e be the set of all numbers of the form $2^e m$, where m is odd. Thus $S_0 = \{1,3,5,7,\ldots\}$, $S_1 = \{2,6,10,14,\ldots\}$, and so on. An arbitrary positive integer n can be written in the form $n = 2^e m$, where m is odd and $e \geq 0$, and thus n is a member of S_e. It follows that the union of the sets S_e for integers $e \geq 0$ is the set of all positive integers. To show that each of these sets S_e is anticlosed, we observe first that each is certainly infinite. Now suppose x and y lie in S_e with $e \geq 0$. We can then write $x = 2^e m$ and $y = 2^e n$ for odd integers m and n, and so $x+y = 2^e(m+n)$. But $m+n$ is not odd, so $x+y$ does not lie in S_e, and thus S_e is anticlosed.

Suppose now that S and T are anticlosed and that every positive integer is in one of these sets. We can assume that 1 lies in S. If also 2 is in S, then $3 = 1+2$ cannot lie in S, and so 3 is in T. But S is infinite, so some number $k > 5$ must lie in S. Since 2 is in S, $k-2 > 2$ and $2+(k-2)$ lies in S, we see that $k-2$ does not lie in S, and thus $k-2$ is in T. Since $k-2 > 3$ and both 3 and $k-2$ are in T, it follows that $3+(k-2) = k+1$ is not in T, and so $k+1$ is in S. However, this is impossible since both 1 and k are in S. We conclude from this contradiction that 2 is not in S, and so 2 must be in T.

Now let k be a member of S with $k > 3$. Then $k+1$ is in T. Furthermore, since 2 is in T, $2+(k-1) = k+1$ is in T and $k-1 > 2$, we see that $k-1$ cannot be in T. Thus $k-1$ is in S. Now 1 is in S and $k-1$ is in S, so $1+(k-1) = k$ cannot be in S, and this is a contradiction. It follows that the set of positive integers is not the union of two anticlosed sets.

5. Susan says that the average number of siblings among all of the children living on her street is three, but that she is a child who has more than three siblings. Find the smallest possible number of families on Susan's street. Justify your answer.

SOLUTION. Suppose that there are n families on Susan's street, and that the ith family has a_i children. Since each child in family i has $a_i - 1$ siblings, it follows that the total number of siblings for all of the children in the ith family is $a_i(a_i-1)$. If we sum the quantities $a_i(a_i-1)$ for $1 \leq i \leq n$ and then divide by the total number of children, we get the average number of siblings, which Susan says is 3. In other words,

$$\frac{a_1(a_1-1) + a_2(a_2-1) + \cdots + a_n(a_n-1)}{a_1 + a_2 + \cdots + a_n} = 3\,.$$

To simplify this equation, multiply both sides by the sum of the a_i, and then subtract 3 times the sum of the a_i from both sides. After doing this, we obtain

$$(a_1^2 - 4a_1) + (a_2^2 - 4a_2) + \cdots + (a_n^2 - 4a_n) = 0\,.$$

Next, observe that $a^2 - 4a + 4 = (a-2)^2$. Thus, if we add 4 to each of the n terms on the left side, and to balance the equation, we add $4n$ to the right side, we obtain n squares which add to $4n$. Also, since Susan has more than three siblings, her family has more than four children, so at least one a_i exceeds 4, and thus the corresponding square $(a_i-2)^2$ exceeds 4.

Now $n \neq 3$ since the only way that 12 can be written as a sum of three squares is $4+4+4$, and similarly, n cannot be 2 or 1. However, 16 can be written as a sum of 4 squares that are not all 4 since $16 = 16+0+0+0$. This corresponds to three families with 2 children each, and one with 6 children. Therefore, the smallest possible number of families is four.