WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2006-2007)

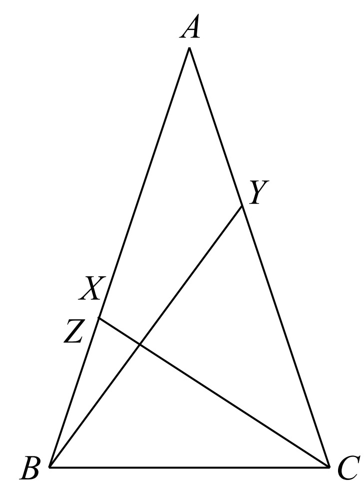
1. Given an integer $n > 2$, let S be the set of all integers m such that $m + n$ is a divisor of $m^2 + n^2$. Show that the set S is finite and that the number of negative numbers in S exceeds the number of positive numbers in S by at least five.

SOLUTION. Of course, $n^2 - m^2 = (n + m)(n - m)$, and so $n + m$ is always a divisor of $n^2 - m^2$. If $n + m$ also divides $n^2 + m^2$, then it divides $(n^2 - m^2) + (n^2 + m^2) = 2n^2$. Conversely, if $n + m$ divides $2n^2$, then it divides $2n^2 - (n^2 - m^2) = n^2 + m^2$. This shows that S is exactly the set of all numbers m such that $n + m$ is a divisor of $2n^2$.

Now let D be the set of all integers (positive and negative) that divide $2n^2$. It follows that S is exactly the set of integers m of the form $m = d - n$, where d is in D. In particular, the number of members of S is the same as the number of members of D, which is finite since $n \neq 0$. The number of positive members of S is exactly the number of members of D exceeding n and the number of negative members of S is the number of members of D that are less than n.

Now if d is in D and exceeds n, then $-d$ is in D and is less than n. In addition there there are at least five members of D that are less than n that we have not yet counted, namely 2, 1, -1, -2 and $-n$. These have not been counted since they are *not* the negatives of numbers exceeding n.

2. In the figure, $\triangle ABC$ is isosceles, with $AB = AC$ and $\angle A = 36°$. Point X on side $\overline{AB}$ and point Y on side $\overline{AC}$ are chosen so that $AX = BC = CY$. Prove that $\overline{BY}$ and $\overline{CX}$ are perpendicular.



SOLUTION. Since $\angle B = \angle C$ and $\angle A = 36°$, it follows that each of $\angle B$ and $\angle C$ is $\frac{1}{2}(180° - 36°) = 72°$. We argue that $\overline{CX}$ is the bisector of $\angle C$. To see this, let $\overline{CZ}$ be the angle bisector, so that $\angle ZCA = \frac{1}{2}(72°) = 36° = \angle A$. Thus $\triangle ZAC$ is isoceles and $AZ = ZC$. Also, $\angle BZC$ is the exterior angle of $\triangle ZAC$ at Z, and hence $\angle BZC = \angle A + \angle ZCA = 36° + 36° = 72° = \angle B$. It follows that $\triangle ZBC$ is isoceles, and $BC = ZC$. Since we saw previously, that $ZC = AZ$, we conclude that $BC = AZ$. Thus the point Z is actually X. This shows that $\overline{CX}$ is the angle bisector, as claimed.

Now $BC = CY$, and thus $\triangle BYC$ is isosceles. But then the angle bisector at C is the same as the altitude to side $\overline{BY}$, so $\overline{BY}$ and $\overline{CX}$ are indeed perpendicular.

3. Find all solutions in positive integers $a < b < c$ to the equation $(a + b + c)^2 = a^3 + b^3 + c^3$.

SOLUTION. Since $0 < a < b < c$ are integers, we have $b \leq c - 1$ and $a \leq c - 2$. Therefore $a + b + c \leq 3c - 3$ and $a^3 + b^3 + c^3 = (a + b + c)^2 \leq (3c - 3)^2 = 9(c - 1)^2$. It follows that

$$a^3 + b^3 \leq 9(c - 1)^2 - c^3 < 9c^2 - c^3 = c^2(9 - c).$$

Since $a^3 + b^3$ is positive, we have $c \leq 8$, and hence $3 \leq c \leq 8$. For the six integers in this range we evaluate $9(c - 1)^2 - c^3$ and obtain the numbers 9, 17, 19, 9, -19 and -71, respectively. But $9(c-1)^2 - c^3 \geq a^3 + b^3$, so $c \neq 7$ and 8. Furthermore, b cannot be as large as 3, since $3^3 = 27$ exceeds these six numbers. Thus $b \leq 2$, so $a = 1$ and $b = 2$. By trying the four remaining possibilities for c, namely $c = 3$, 4, 5 and 6, we see that only $c = 3$ satisfies the given equation with these values of a and b. In other words, the unique solution is $a = 1$, $b = 2$ and $c = 3$.

4. Suppose that for each integer $k \geq 1$, we have an unlimited supply of rectangular $2 \times k$ tiles. Given an integer $n \geq 1$, write $a(n)$ to denote the number of ways that a $2 \times n$ rectangle can be covered using our tiles. It is clear, for example, that $a(1) = 1$, and a little experimentation shows that $a(2) = 3$ and $a(3) = 6$. Compute $a(7)$.

SOLUTION. Suppose that the given $2 \times n$ rectangle is oriented so that the sides of length n are horizontal, and consider the tile or tiles touching the left edge of the rectangle. First consider the case where this left edge touches just one tile, say a $2 \times k$ tile, with its side of length k oriented horizontally. In that case, after the $2 \times k$ tile is placed, there remains a $2 \times (n-k)$ rectangle still to be covered, so there are $a(n-k)$ ways to complete the job. Of course, if $k = n$, there is just one way to complete the job, do nothing. Thus, it is convenient to define $a(0) = 1$. Now $1 \leq k \leq n$, and so after the leftmost tile is placed, the total number of ways that the tiling can be completed is $a(n-1) + a(n-2) + \cdots + a(1) + a(0)$, corresponding to $k = 1$, $k = 2$, and so on, through $k = n$.

We have not yet counted all $a(n)$ possible ways to tile our $2 \times n$ rectangle because the left edge may be touching *two* tiles, each of size 2×1, with the sides of length 2 oriented horizontally. This leaves a $2 \times (n-2)$ rectangle still to be covered, and this can be done in $a(n-2)$ ways. Our complete count is therefore

$$a(n) = a(n-1) + 2a(n-2) + a(n-3) + \cdots + a(2) + a(1) + a(0)\,.$$

If we subtract this from the corresponding formula for $a(n+1)$, we get $a(n+1) - a(n) = a(n) + a(n-1) - a(n-2)$, and thus $a(n+1) = 2a(n) + a(n-1) - a(n-2)$ for $n \geq 2$. We know that $a(2) = 3$, $a(1) = 1$ and $a(0) = 1$. Thus $a(3) = 2a(2) + a(1) - a(0) = 6 + 1 - 1 = 6$, $a(4) = 2(6) + 3 - 1 = 14$ and $a(5) = 2(14) + 6 - 3 = 31$. Continuing like this, we get $a(6) = 2(31) + 14 - 6 = 70$ and finally, $a(7) = 2(70) + 31 - 14 = 157$.

5. Find all pairs of positive numbers x and y such that $x^3 - y^3 = 100$ and both $x - y$ and xy are integers.

SOLUTION. Write $n = x - y$ and $m = xy$, so that n and m are integers. Since $x^3 - y^3 = 100$, we see that $x > y$, and thus n is positive. Also, since x and y are positive, it follows that $m = xy$ is positive. We have

$$100 = x^3 - y^3 = (y+n)^3 - y^3 = 3ny^2 + 3n^2y + n^3 = 3ny(y+n) + n^3\,,$$

and because $y + n = x$ and $xy = m$, this yields $100 = 3mn + n^3 > n^3$. Since $5^3 = 125$ exceeds 100, we see that $n \leq 4$. Also, $100 = n(3m + n^2)$, and so n is a divisor of 100, and thus $n \neq 3$. If $n = 2$, then $50 = 3m + 4$, and $m = 46/3$, which is not an integer. The surviving possibilities, therefore, are $n = 1$ and $n = 4$, and we get $m = 33$ and $m = 3$, respectively.

If $n = 1$, the equation $3ny^2 + 3n^2y + n^3 - 100 = 0$ becomes $3y^2 + 3y - 99 = 0$, and so $y^2 + y - 33 = 0$. The quadratic formula yields $y = (-1 \pm \sqrt{133})/2$, and since $y > 0$, we have $y = (\sqrt{133} - 1)/2$ and $x = y + 1 = (\sqrt{133} + 1)/2$. It is easy to check that $x^3 - y^3 = 100$ and $xy = 33$, as wanted. If $n = 4$, our quadratic equation becomes $12y^2 + 48y - 36 = 0$, and so $y^2 + 4y - 3 = 0$. This yields $y = (-4 \pm \sqrt{28})/2 = -2 \pm \sqrt{7}$. Because $y > 0$, we must have $y = \sqrt{7} - 2$ and $x = y + 4 = \sqrt{7} + 2$. We again check that $x^3 - y^3 = 100$ and $xy = 3$, as required.