1. Given an integer \( n > 2 \), let \( S \) be the set of all integers \( m \) such that \( m + n \) is a divisor of \( m^2 + n^2 \). Show that the set \( S \) is finite and that the number of negative numbers in \( S \) exceeds the number of positive numbers in \( S \) by at least five.

**SOLUTION.** Of course, \( n^2 - m^2 = (n + m)(n - m) \), and so \( n + m \) is always a divisor of \( n^2 - m^2 \). If \( n + m \) also divides \( n^2 + m^2 \), then it divides \( (n^2 - m^2) + (n^2 + m^2) = 2n^2 \). Conversely, if \( n + m \) divides \( 2n^2 \), then it divides \( 2n^2 - (n^2 - m^2) = n^2 + m^2 \). This shows that \( S \) is exactly the set of all numbers \( m \) such that \( n + m \) is a divisor of \( 2n^2 \).

Now let \( D \) be the set of all integers (positive and negative) that divide \( 2n^2 \). It follows that \( S \) is exactly the set of integers \( m \) of the form \( m = d - n \), where \( d \) is in \( D \). In particular, the number of members of \( S \) is the same as the number of members of \( D \), which is finite since \( n \neq 0 \). The number of positive members of \( S \) is exactly the number of members of \( D \) exceeding \( n \) and the number of negative members of \( S \) is the number of members of \( D \) that are less than \( n \).

Now if \( d \) is in \( D \) and exceeds \( n \), then \( -d \) is in \( D \) and is less than \( n \). In addition there are at least five members of \( D \) that are less than \( n \) that we have not yet counted, namely 2, 1, \(-1\), \(-2\) and \(-n\). These have not been counted since they are *not* the negatives of numbers exceeding \( n \).

2. In the figure, \( \triangle ABC \) is isosceles, with \( AB = AC \) and \( \angle A = 36^\circ \). Point \( X \) on side \( AB \) and point \( Y \) on side \( AC \) are chosen so that \( AX = BC = CY \). Prove that \( BY \) and \( CX \) are perpendicular.

**SOLUTION.** Since \( \angle B = \angle C \) and \( \angle A = 36^\circ \), it follows that each of \( \angle B \) and \( \angle C \) is \( \frac{1}{2}(180^\circ - 36^\circ) = 72^\circ \). We argue that \( CX \) is the bisector of \( \angle C \). To see this, let \( CZ \) be the angle bisector, so that \( \angle ZCA = \frac{1}{2}(72^\circ) = 36^\circ = \angle A \). Thus \( \triangle ZAC \) is isosceles and \( AZ = ZC \). Also, \( \angle BZC \) is the exterior angle of \( \triangle ZAC \) at \( Z \), and hence \( \angle BZC = \angle A + \angle ZCA = 36^\circ + 36^\circ = 72^\circ = \angle B \). It follows that \( \triangle ZBC \) is isosceles, and \( BC = ZC \). Since we saw previously, that \( ZC = AZ \), we conclude that \( BC = AZ \). Thus the point \( Z \) is actually \( X \). This shows that \( CX \) is the angle bisector, as claimed.

Now \( BC = CY \), and thus \( \triangle BYC \) is isosceles. But then the angle bisector at \( C \) is the same as the altitude to side \( BY \), so \( BY \) and \( CX \) are indeed perpendicular.

3. Find all solutions in positive integers \( a < b < c \) to the equation \((a + b + c)^2 = a^3 + b^3 + c^3\).

**SOLUTION.** Since \( 0 < a < b < c \) are integers, we have \( b \leq c - 1 \) and \( a \leq c - 2 \). Therefore \( a + b + c \leq 3c - 3 \) and \( a^3 + b^3 + c^3 = (a + b + c)^2 \leq (3c - 3)^2 = 9(c - 1)^2 \). It follows that

\[
a^3 + b^3 \leq 9(c - 1)^2 - c^3 < 9c^2 - c^3 = c^2(9 - c).
\]

Since \( a^3 + b^3 \) is positive, we have \( c \leq 8 \), and hence \( 3 \leq c \leq 8 \). For the six integers in this range we evaluate \( 9(c - 1)^2 - c^3 \) and obtain the numbers 9, 17, 19, 9, -19 and -71, respectively. But \( 9(c - 1)^2 - c^3 \geq a^3 + b^3 \), so \( c \neq 7 \) and 8. Furthermore, \( b \) cannot be as large as 3, since \( 3^3 = 27 \) exceeds these six numbers. Thus \( b \leq 2 \), so \( a = 1 \) and \( b = 2 \). By trying the four remaining possibilities for \( c \), namely \( c = 3, 4, 5 \) and 6, we see that only \( c = 3 \) satisfies the given equation with these values of \( a \) and \( b \). In other words, the unique solution is \( a = 1, b = 2 \) and \( c = 3 \).
4. Suppose that for each integer \( k \geq 1 \), we have an unlimited supply of rectangular \( 2 \times k \) tiles. Given an integer \( n \geq 1 \), write \( a(n) \) to denote the number of ways that a \( 2 \times n \) rectangle can be covered using our tiles. It is clear, for example, that \( a(1) = 1 \), and a little experimentation shows that \( a(2) = 3 \) and \( a(3) = 6 \). Compute \( a(7) \).

**SOLUTION.** Suppose that the given \( 2 \times n \) rectangle is oriented so that the sides of length \( n \) are horizontal, and consider the tile or tiles touching the left edge of the rectangle. First consider the case where this left edge touches just one tile, say a \( 2 \times k \) tile, with its side of length \( k \) oriented horizontally. In that case, after the \( 2 \times k \) tile is placed, there remains a \( 2 \times (n-k) \) rectangle still to be covered, so there are \( a(n-k) \) ways to complete the job. Of course, if \( k = n \), there is just one way to complete the job, do nothing. Thus, it is convenient to define \( a(0) = 1 \). Now \( 1 \leq k \leq n \), and so after the leftmost tile is placed, the total number of ways that the tiling can be completed is \( a(n-1) + a(n-2) + \cdots + a(1) + a(0) \), corresponding to \( k = 1, k = 2 \), and so on, through \( k = n \).

We have not yet counted all \( a(n) \) possible ways to tile our \( 2 \times n \) rectangle because the left edge may be touching *two* tiles, each of size \( 2 \times 1 \), with the sides of length 2 oriented horizontally. This leaves \( 2 \times (n-2) \) rectangle still to be covered, and this can be done in \( a(n-2) \) ways. Our complete count is therefore

\[
a(n) = a(n-1) + 2a(n-2) + a(n-3) + \cdots + a(2) + a(1) + a(0).\]

If we subtract this from the corresponding formula for \( a(n+1) \), we get \( a(n+1) - a(n) = a(n) + a(n-1) - a(n-2) \), and thus \( a(n+1) = 2a(n) + a(n-1) - a(n-2) \) for \( n \geq 2 \). We know that \( a(2) = 3 \), \( a(1) = 1 \) and \( a(0) = 1 \). Thus \( a(3) = 2a(2) + a(1) - a(0) = 6 + 1 - 1 = 6 \), \( a(4) = 2(6) + 3 - 1 = 14 \) and \( a(5) = 2(14) + 6 - 3 = 31 \). Continuing like this, we get \( a(6) = 2(31) + 14 - 6 = 70 \) and finally, \( a(7) = 2(70) + 31 - 14 = 157 \).

5. Find all pairs of positive numbers \( x \) and \( y \) such that \( x^3 - y^3 = 100 \) and both \( x - y \) and \( xy \) are integers.

**SOLUTION.** Write \( n = x - y \) and \( m = xy \), so that \( n \) and \( m \) are integers. Since \( x^3 - y^3 = 100 \), we see that \( x > y \), and thus \( n \) is positive. Also, since \( x \) and \( y \) are positive, it follows that \( m = xy \) is positive. We have

\[
100 = x^3 - y^3 = (y + n)^3 - y^3 = 3ny^2 + 3n^2y + n^3 = 3ny(y + n) + n^3,
\]

and because \( y + n = x \) and \( xy = m \), this yields \( 100 = 3mn + n^3 > n^3 \). Since \( 5^3 = 125 \) exceeds 100, we see that \( n \leq 4 \). Also, \( 100 = n(3m + n^2) \), and so \( n \) is a divisor of 100, and thus \( n \neq 3 \). If \( n = 2 \), then \( 50 = 3m + 4 \), and \( m = 46/3 \), which is not an integer. The surviving possibilities, therefore, are \( n = 1 \) and \( n = 4 \), and we get \( m = 33 \) and \( m = 3 \), respectively.

If \( n = 1 \), the equation \( 3ny^2 + 3n^2y + n^3 - 100 = 0 \) becomes \( 3y^2 + 3y - 99 = 0 \), and so \( y^2 + y - 33 = 0 \). The quadratic formula yields \( y = (-1 \pm \sqrt{133})/2 \), and since \( y > 0 \), we have \( y = (\sqrt{133} - 1)/2 \) and \( x = y + 1 = (\sqrt{133} + 1)/2 \). It is easy to check that \( x^3 - y^3 = 100 \) and \( xy = 33 \), as wanted. If \( n = 4 \), our quadratic equation becomes \( 12y^2 + 48y - 36 = 0 \), and so \( y^2 + 4y - 3 = 0 \). This yields \( y = (-4 \pm \sqrt{28})/2 = -2 \pm \sqrt{7} \). Because \( y > 0 \), we must have \( y = \sqrt{7} - 2 \) and \( x = y + 4 = \sqrt{7} + 2 \). We again check that \( x^3 - y^3 = 100 \) and \( xy = 3 \), as required.