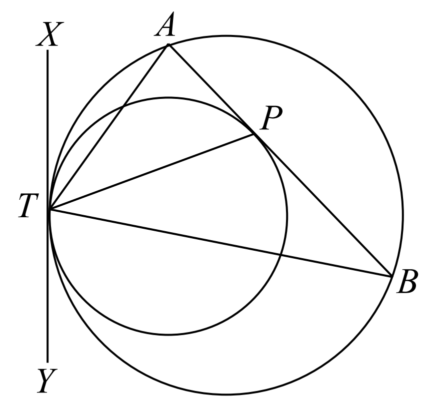
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2006-2007)

1. Prove that $(x+1)(x^2+1)(x^3+1) \le 4(x^6+1)$ for all real numbers x.

SOLUTION. We first note that $x^6 + 1 = (x^2)^3 + 1$, and consequently this polynomial factors as $(x^2+1)(x^4-x^2+1)$. Therefore, since x^2+1 is positive for all real numbers x, we can prove the desired inequality if we show that $(x+1)(x^3+1) \le 4(x^4-x^2+1)$ for all real x. In other words, our task is to prove that $4(x^4-x^2+1)-(x+1)(x^3+1) \ge 0$, or equivalently, $3x^4-x^3-4x^2-x+3 \ge 0$. Next, we attempt to factor the polynomial $3x^4-x^3-4x^2-x+3$. If we substitute $x = 1$, the result is 0, and so $x-1$ is a factor. By polynomial long division, we get $3x^4-x^3-4x^2-x+3 = (x-1)(3x^3+2x^2-2x-3)$. The second factor also becomes zero when we substitute $x = 1$, so it too has a factor of $x-1$, and by long division we obtain $3x^4-x^3-4x^2-x+3 = (x-1)^2(3x^2+5x+3)$. Since $(x-1)^2 \ge 0$ for all real x, our problem reduces to showing that $3x^2+5x+3 \ge 0$ for all x. One way to see this is to observe that this quadratic polynomial is positive when $x = 0$ and that it is never zero because the discriminant $5^2 - 4(3)(3)$ is negative. Its graph, therefore, never crosses the x-axis, and so $3x^2+5x+3 > 0$ for all x. This completes the proof.

2. In the diagram, the two circles are tangent at point T, and chord $\overline{AB}$ of the large circle is tangent to the small circle at point P. Prove that $\overline{TP}$ bisects $\angle ATB$.



SOLUTION. Draw the common tangent line $\overline{XY}$ as shown and recall that the angle between a chord and tangent to a circle is equal in degrees to half the arc they subtend. In particular, $\angle XTA$ is half of arc $\overset{\frown}{AT}$. Now, inscribed angle $\angle TBA$ also subtends $\overset{\frown}{AT}$ and so it too is equal in degrees to half of that arc. It follows that $\angle XTA = \angle TBA$. Next, notice that $\angle YTP = \angle BPT$ because two tangents to a circle make equal angles with the chord joining the points of tangency. We now see that

$$\begin{aligned}\angle ATP &= 180^\circ - \angle XTA - \angle YTP \\ &= 180^\circ - \angle TBA - \angle BPT = \angle BTP,\end{aligned}$$

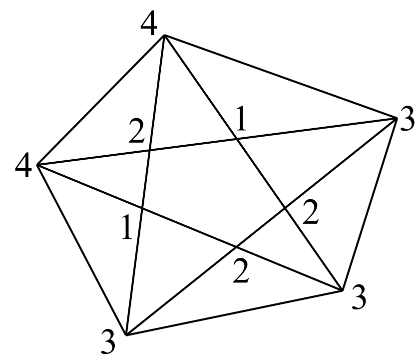
where the last equality holds because the sum of the angles in $\triangle PTB$ is 180°. It follows that $\overline{TP}$ bisects $\angle ATB$, as wanted.

3. Let $a < b < c$ be positive integers such that no prime number is a divisor of all three of them. Suppose N is a positive integer that is divisible by each of a, b, and c. Prove that $N > a^{3/2}$.

SOLUTION. For each prime p, let a_p, b_p, c_p, and N_p be the exponent of p in the prime factorizations of a, b, c, and N respectively. In other words, $a = \prod_p p^{a_p}$ is the product of the factors p^{a_p} over all primes p, and similarly for b, c, and N. Since N is divisible by each of a, b, and c, it follows that N_p is greater than or equal to each of a_p, b_p, and c_p. Thus, $N_p \ge \max\{a_p, b_p, c_p\}$. Furthermore, since a, b, and c are not all divisible by the prime p, at least one of a_p, b_p, or c_p is

0. This implies that $2 \cdot \max\{a_p, b_p, c_p\} \geq a_p + b_p + c_p$. We conclude that the exponent of p in the prime factorization of N^2, namely $2 \cdot N_p$, is greater than or equal to the exponent of p in the prime factorization of abc, namely $a_p + b_p + c_p$. Since this is true for all primes p, it follows that N^2 is divisible by abc, and hence $abc \leq N^2$. But $a < b < c$, so we know that $a^3 < abc \leq N^2$, and this yields the desired result.

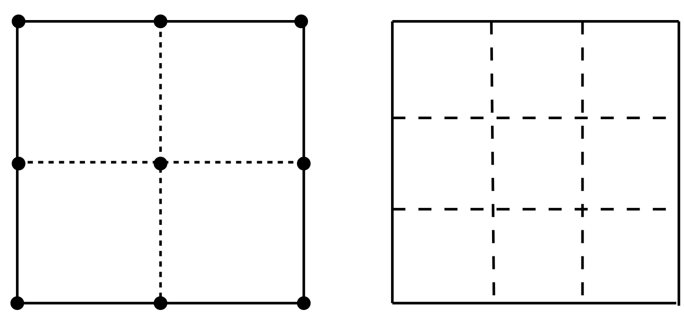
4. The five diagonals of a pentagon intersect at ten points (including the five vertices of the pentagon). Find all positive integers n such that the ten numbers 1, 1, 2, 2, 2, 3, 3, 3, 4, and n can be placed on these ten intersection points in such a way that the sums of the four numbers along the diagonals are all equal.



SOLUTION. Assume that the ten numbers 1, 1, 2, 2, 2, 3, 3, 3, 4, and n are placed at the ten diagonal intersection points in such a way that the sum of the four numbers on each diagonal is s. We add together these sums for each of the five diagonals and observe that each of the ten numbers is counted twice. This yields $5s = 2n+42$. Consequently, s is even, and we have $s = 2k$ and $n = 5k - 21$ for some integer k. On the one hand, we have $k \geq 5$, since $n > 0$. On the other hand, the numbers on one of the diagonals containing n add up to at least $n+1+1+2 = 5k-17$, which gives $2k = s \geq 5k - 17$. Thus $k \leq 17/3$, so the only possibility is $k = 5$, which corresponds to $n = 4$ and $s = 10$.

It is indeed possible to arrange the ten numbers 1, 1, 2, 2, 2, 3, 3, 3, 4, and 4 in the desired manner. For example, one can place the two 4s at adjacent vertices, as indicated, and the three 3s at the remaining three vertices. The 2s are placed at the points on the inside pentagon that are either opposite to or adjacent to both 4s. Finally, the 1s are placed in the two remaining locations.

5. What is the largest number of points that can be placed in (or on the boundary of) a 2×2 square so that the distance between each pair of points is at least 1? Justify your answer.



SOLUTION. We show that the largest number of such points is 9. To start with, let us divide the 2×2 square into four equal pieces, as indicated in the left-hand diagram. Then each of the smaller squares is size 1×1 and we choose the nine corner points of these four squares. Obviously, the distance between each pair of these points is at least 1, and therefore we have shown that 9 points are possible.

On the other hand, let us now divide the 2×2 square into nine equal pieces, as indicated in the right-hand diagram. Then each of the smaller squares is size $(2/3) \times (2/3)$, and if we place 10 or more points in the larger square, then at least two of them will have to be contained in (or on the boundary of) one of the smaller squares. In particular, the distance between these two points will be at most the length of a diagonal of the small square, namely $2\sqrt{2}/3$. But $2\sqrt{2}/3 < 1$, so we have shown that if 10 or more points are placed in the 2×2 square, then at least one pair of points will be at a distance < 1 apart. In other words, 9 is the largest possibility.