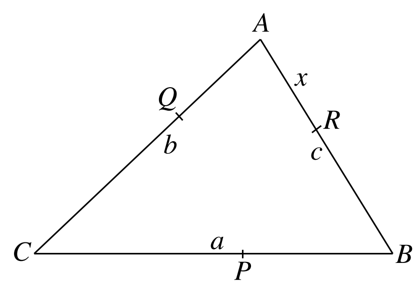
**WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH**
**SOLUTIONS TO PROBLEM SET II (2006-2007)**

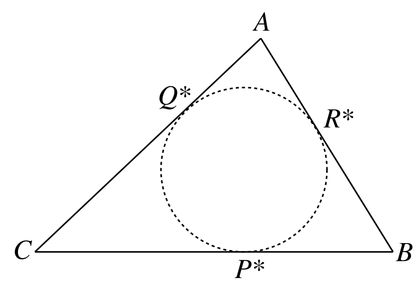
**1.** It is a fact that $2^{38} = 274877906944$, which is a number having its two rightmost digits equal. Does there exist some power of 2 whose three rightmost digits are equal? Does there exist some power of 2 whose four rightmost digits are equal? Prove that your answers are correct.

**SOLUTION.** Since $2^{38}$ ends in $\ldots 06944$, we see that $2^{39} = 2^{38}\cdot 2$ ends in $\ldots 3888$, and therefore it is possible to have a power of 2 where the three rightmost digits are equal. To show that the four rightmost digits of a power of 2 cannot be equal, we assume that $2^n$ ends with four copies of the digit $d$, and we derive a contradiction. Observe that $d \neq 0$ since $2^n$ is not a multiple of 5. Now $2^n - 1111\cdot d$ ends with four zeros, and so it is a multiple of 10000. Thus $2^n = 10000\cdot k + 1111\cdot d$, for some integer $k \geq 0$. Since $2^n \geq 1111$, we certainly have $n \geq 4$, and hence $2^n$ is a multiple of 16. Since 10000 is also a multiple of 16, it follows that $1111\cdot d$ must be a multiple of 16. But 1111 is an odd number, and $0 < d \leq 9$, so we have a contradiction.

**2.** Points $P$, $Q$ and $R$ lie on the sides of $\triangle ABC$ as shown, with $P$ on $\overline{BC}$, $Q$ on $\overline{AC}$ and $R$ on $\overline{AB}$. These three points are positioned so that $AQ = AR$, $BR = BP$ and $CP = CQ$. Prove that the inscribed circle of $\triangle ABC$ passes through points $P$, $Q$ and $R$.




**SOLUTION.** Let $BC = a$, $CA = b$, $AB = c$, and set $x = AR$. Then $RB = AB - x = c - x$, and since $RB = BP$, we have $BP = c - x$. It follows that $PC = a - (c - x) = a - c + x$, so $CQ = a - c + x$ and hence $QA = b - (a - c + x) = b - a + c - x$. But $QA = AR = x$, so $x = b - a + c - x$ and we deduce that $x = (b - a + c)/2$. In particular, the distance $x$ from $A$ to $R$ is completely determined by the lengths of the sides of $\triangle ABC$, and thus the position of $R$ on side $\overline{AB}$ is determined. Similarly, the positions of $Q$ on side $\overline{AC}$ and of $P$ on side $\overline{BC}$ are uniquely determined.




Now draw the inscribed circle of $\triangle ABC$ and let $P^*$, $Q^*$ and $R^*$ be its points of tangency with sides $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ respectively. Since two tangents drawn to a circle from an outside point are equal, it follows that $AQ^* = AR^*$, $BR^* = BP^*$ and $CP^* = CQ^*$. In other words, the points $P^*$, $Q^*$ and $R^*$ satisfy the conditions assumed for $P$, $Q$ and $R$. Since we showed that there is at most one way to choose such points, it follows that $P^* = P$, $Q^* = Q$ and $R^* = R$. In other words, the inscribed circle goes through points $P$, $Q$ and $R$, as wanted.

**3.** How many ten letter "words" are there like XXYXYYXYXX, which are composed of Xs and Ys, and which contain neither three consecutive Xs nor three consecutive Ys.

**SOLUTION.** Let $F(n)$ denote the number of $n$-letter words consisting of Xs and Ys and not containing three consecutive identical letters. Thus $F(1) = 2$ since the one-letter words X and Y meet the no-triple-repeat condition. Since all four two-letter possibilities XX, XY, YX and YY meet the condition, we have $F(2) = 4$.

Assuming that $n \geq 3$, how can we construct an $n$-letter word with no triple repeats? One way would be to start with an $(n-2)$-letter word $w$ and to append either XX if $w$ ends in Y, or YY if $w$ ends in X. Since there are $F(n-2)$ possibilities for $w$, this allows us to construct $F(n-2)$

words of length $n$, and these are all of the $n$-letter words with no triple repeats and having a repeated letter at the end. Similarly, we can start with an $(n-1)$-letter word $v$ and append either X if $v$ ends in Y, or Y if $v$ ends in X. This yields $F(n-1)$ acceptable words, and these are all the acceptable $n$-letter words whose last two letters are different. We thus have a total of $F(n-2)+F(n-1)$ $n$-letter words with no triple repeats, and thus $F(n)=F(n-2)+F(n-1)$. Now $F(1)=2$ and $F(2)=4$, so $F(3)=2+4=6$, $F(4)=4+6=10$, $F(5)=6+10=16$, $F(6)=10+16=26$, $F(7)=16+26=42$, $F(8)=26+42=68$, $F(9)=42+68=110$ and finally, $F(10)=68+110=178$.

**4.** It is easy to check that

$$\frac{1}{8}=\frac{1}{3^2}+\frac{1}{12^2}+\frac{1}{15^2}+\frac{1}{20^2}\,,$$

and so $1/8$ is the sum of the reciprocals of four different square integers. Decide whether or not it is possible to write $1/8$ as the sum of the reciprocals of *three* different square integers. Prove that your answer is correct.

**SOLUTION.** Suppose that $1/8=1/a^2+1/b^2+1/c^2$, where $a$, $b$ and $c$ are distinct positive integers. By renaming $a$, $b$ and $c$ if necessary, we can assume that $a<b<c$, and thus $1/c^2<1/b^2<1/a^2$. It follows that $1/8=1/a^2+1/b^2+1/c^2<3/a^2$, and hence $a^2<24$. Also, $1/8>1/a^2$, and so $a^2>8$. This yields just two possibilities, namely $a^2=9$ or $a^2=16$.

Suppose first that $a^2=9$. Then $1/b^2+1/c^2=1/8-1/9=1/72$. Also, $1/c^2<1/b^2$, so $1/72<2/b^2$ and $b^2<144$. Since $1/72>1/b^2$, we have $b^2>72$, and hence the only possibilities are $b^2=81$, $b^2=100$ or $b^2=121$. Since $1/c^2=1/72-1/b^2$, we have $c^2=72b^2/(b^2-72)$. But none of the quantities obtained by substituting 81, 100 or 121 for $b^2$ in $72b^2/(b^2-72)$ is a square integer, and so we get no solution with $a^2=9$.

The only remaining possibility is $a^2=16$, and in this case, $1/b^2+1/c^2=1/8-1/16=1/16$. Reasoning as before, we have $1/16<2/b^2$, so $b^2<32$. Also, $b^2>16$, so $b^2=25$ and $c^2=16b^2/(b^2-16)=16{\cdot}25/9$. Since $c^2$ is not even an integer here, we have proved that $1/8$ cannot be written as a sum of reciprocals of three different square integers.

**5.** Let $n$ be a positive integer and let $a$, $b$ and $c$ be real numbers. Suppose that for every integer $m$, the quantity

$$\frac{1}{n}m^3+am^2+bm+c$$

is an integer. Prove that $n$ must be one of the numbers 1, 2, 3 or 6.

**SOLUTION.** Since $(1/n)m^3+am^2+bm+c$ is an integer for every integer $m$, the same is true for $(1/n)(m+1)^3+a(m+1)^2+b(m+1)+c$, and thus the difference between these quantities is also an integer for every integer $m$. This difference is equal to

$$\frac{1}{n}\left((m+1)^3-m^3\right)+a\left((m+1)^2-m^2\right)+b((m+1)-m)+(c-c)=\frac{3}{n}m^2+dm+e\,,$$

for some real constants $d$ and $e$. In other words, we have found a quadratic expression of the form $(3/n)m^2+dm+e$ which we know is an integer for every integer $m$.

Now we use the same trick again. We know that $(3/n)m^2+dm+e$ is an integer for every integer $m$, and so the same is true for $(3/n)(m+1)^2+d(m+1)+e$. The difference of these quantities is therefore an integer for all integers $m$, and this difference is $(3/n)((m+1)^2-m^2)+d$, which we can write as $(6/n)m+f$, for some real constant $f$. If we apply this trick once again, we see that $(6/n)((m+1)-m)=6/n$ is an integer for all integers $m$. Thus $6/n$ is an integer and, since $n$ is a positive integer, the only possibilities are $n=1$, 2, 3 or 6.