WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2006-2007)

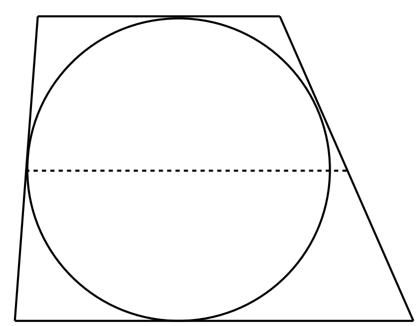
1. Find all possibilities for real numbers a and b such that the polynomial x^2+ax+b has values between -2 and 2 inclusive, for all x with $0 \le x \le 4$.

SOLUTION. For simplicity of notation, write $f(x) = x^2 + ax + b$. Since $f(x)$ lies between -2 and 2 for all x with $0 \le x \le 4$, it follows that if x and y are two numbers in the interval from 0 to 4 inclusive, then $f(x) - f(y) \le 4$. In particular, $f(4) - f(2) \le 4$ and $f(0) - f(2) \le 4$. Now $f(4) - f(2) = (16 + 4a + b) - (4 + 2a + b) = 12 + 2a$, and this yields $12 + 2a \le 4$, from which we deduce that $2a \le -8$ and $a \le -4$. Also, $f(0) - f(2) = b - (4 + 2a + b) = -4 - 2a$, and thus $-4 - 2a \le 4$. We deduce that $-2a \le 8$, and thus $a \ge -4$. We now know that $a \le -4$ and also that $a \ge -4$, and so we have $a = -4$.

Now $f(0) = b$, and so $b \le 2$. Also, since $a = -4$, we can compute that $f(2) = b - 4$, and so $b - 4 \ge -2$ and $b \ge 2$. Since $b \le 2$ and $b \ge 2$, we conclude that $b = 2$.

To complete the solution, we must check that the polynomial $f(x) = x^2 - 4x + 2$ actually satisfies the condition that its values for $0 \le x \le 4$ lie between -2 and 2, inclusive. We can write $f(x) = (x - 2)^2 - 2$, and we see that $f(x) \ge -2$ for all real x. The largest values of $f(x)$ for $0 \le x \le 4$ occur when $(x - 2)^2$ is as large as possible, namely when $x = 0$ and $x = 4$, and since $f(0) = 2 = f(4)$, it follows that $f(x) \le 2$ for all x in the interval.

2. Each side of a trapezoid is tangent to a circle of radius 1, as shown. Prove that the area of the trapezoid is at least 4. (In fact, this conclusion holds even if the given quadrilateral is not a trapezoid, but the proof seems harder in that case.)



SOLUTION. Since the two parallel sides of the trapezoid are perpendicular to a diameter of the circle, which has length 2, we see that the distance between these two sides is 2, and this is the "height" of the trapezoid. The median line of the trapezoid, which is the line segment joining the midpoints of the two non-parallel sides, is half way between the parallel sides, and hence it passes through the center of the circle. The ends of this median line segment lie outside (or on) the circle, and so the median contains a diameter of the circle. The length of the median is therefore at least 2. Since the area of a trapezoid is the product of the height and the length of the median, it follows that the area is at least 4.

3. Let f be a function defined for all nonzero real numbers. (Recall that this means that for each such number x, there is a uniquely determined real number $f(x)$.) Suppose that $f(1/x) + 2f(x) = x$ for all nonzero real x. Determine the function f.

SOLUTION. Since the equation $f(1/x)+2f(x) = x$ holds for all nonzero numbers x, it remains valid if we substitute $1/x$ for x. Since $1/(1/x) = x$, this yields the new equation $f(x)+2f(1/x) = 1/x$. Now multiply both sides of the original equation by 2 and then subtract the new equation from it. We obtain $3f(x) = 2x - (1/x) = (2x^2 - 1)/x$, and thus $f(x) = (2x^2 - 1)/(3x)$.

To check that this function really does satisfy the original equation, we compute that

$$f(1/x) = \frac{(2/x^2) - 1}{(3/x)} = \frac{2 - x^2}{3x}\,.$$

Now adding $2f(x)$, we get $(3x^2)/(3x) = x$, as required.

4. Find all prime numbers of the form $100\cdots001$, where the total number of zeros between the first and last digits is even.

SOLUTION. First, we recall a basic factorization fact: if m is odd, then

$$a^m + b^m = (a+b)(a^{m-1} - a^{m-2}b + a^{m-3}b^2 - a^{m-4}b^3 + \cdots + b^{m-1})\,,$$

where the signs in the second factor alternate and the sign of the last term is "+" because m is odd. To see why this is true, observe that multiplication of the second factor by a yields all of the terms a^ib^j such that $i + j = m$, with $i > 0$ and $j \geq 0$. Furthermore, the term a^ib^j occurs with a "+" sign precisely when i is odd. Similarly, multiplication of the second factor by b yields all of the terms a^ib^j such that $i + j = m$, with $i \geq 0$ and $j > 0$. Here, a^ib^j occurs with a "+" when j is odd. If $i + j = m$, which is odd, then exactly one of i or j is odd, and thus a^ib^j occurs with opposite signs in a times the second factor and b times the second factor. When these are added, everything cancels except a^m which appears only in the first product and b^m, which appears only in the second product.

The number $n = 100\cdots001$, with exactly k zeros between the first and last digits is $10^{k+1}+1 = 10^{k+1}+1^{k+1}$. If k is even, then $m = k+1$ is odd, and so by the factorization formula of the previous paragraph, we see that $10+1 = 11$ must divide n. It follows that n cannot be prime unless $n = 11$, which is the case $k = 0$. Thus 11 is the unique prime satisfying the condition of the problem.

5. Eight lamps, each with an on-off switch, are arranged in a circle. A lamplighter can flip (in other words, change the state of) the switches, but he is not allowed to flip just one at a time. When he switches lamps on or off, he is required to flip the switches of three consecutive lamps simultaneously. (For example, he can flip the switches of lamps 7, 8 and 1 at the same time.) Prove that no matter what set of lamps was turned on at the start, the lamplighter can turn all the lamps on.

SOLUTION. To prove that the lamplighter can turn all the lamps on regardless of the starting configuration, it suffices to find a procedure for changing the state of any one lamp while leaving all of the others unchanged. The lamplighter can then apply this procedure for each of the lamps that was originally off, and the result will be that all eight lamps are on.

Suppose the lamplighter flips the switches on lamps 1, 2 and 3, then in turn on lamps 4, 5 and 6, lamps 7, 8 and 1, lamps 2, 3 and 4 and finally on lamps 5, 6 and 7. The result is that the switch on each lamp other than 8 was flipped twice, and so those lamps are in their original on-off state. But the switch on lamp 8 was flipped just once, and so that lamp has changed its state. A similar procedure allows the lamplighter to change the state of any arbitrarily selected lamp.