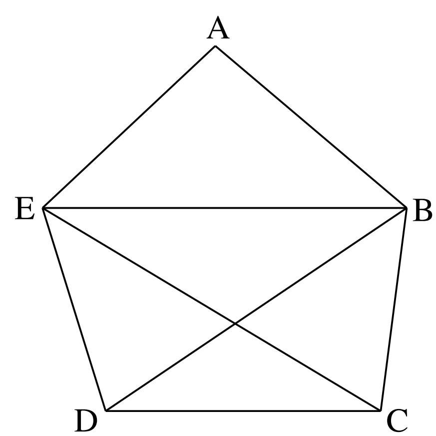
WISCONSIN MATHEMATICS, SCIENCE AND ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET V (2005-2006)

1. A club with seven members wants to form a number of three-person committees, but they require that no two committees should have more than one member in common. For example, if the members of the club are A, B, C, D, E, F and G, they could form these five committees: {A,B,G}, {A,C,F}, {A,D,E}, {B,C,E} and {D,F,G}. What is the maximum number of committees the club can form? Prove that your answer is correct.

SOLUTION. Since no two committees can have more than one member in common, it follows that each set of two of the seven club members can be together on at most one committee. Also, observe that the number of ways to choose two of the seven club members is $(7{\cdot}6)/2 = 21$. Since a committee of three people contains three pairs of people, and we have only 21 pairs in total, it follows that there cannot be more than $21/3 = 7$ committees.

In fact, it is possible to set up seven committees such that no two of these committees have more than one member in common. (But note that if we start with the five committees given in the example, it is not possible to create two more committees to form a set of seven with the desired overlap property.) If as in the example, the seven club members are A, B, C, D, E, F and G, it is easy to check that these seven committees work: $\{A, B, F\}$, $\{A, C, E\}$, $\{A, D, G\}$, $\{B, C, D\}$, $\{B, E, G\}$, $\{C, F, G\}$ and $\{D, E, F\}$. In fact, every two of these committees overlap in exactly one person.

2. Let us say that a diagonal of a pentagon is *good* if it is parallel to one of the sides of the pentagon. Show that if four of the five diagonals of a pentagon are good, then the fifth diagonal is also good.



SOLUTION. Draw the diagonals $\overline{CE}$ and $\overline{BD}$, as indicated, and notice that $\triangle CDE$ and $\triangle BCD$ share a common base $\overline{CD}$. Thus the two triangles have the same area if and only if the altitudes from E to $\overline{CD}$ and from B to $\overline{CD}$ have equal length and hence if and only if $\overline{EB}$ is parallel to $\overline{CD}$. In other words, the diagonal $\overline{EB}$ is good if and only if the areas of the two "opposite" triangles $\triangle CDE$ and $\triangle BCD$ are equal. In particular, if the four diagonals $\overline{AC}$, $\overline{BD}$, $\overline{CE}$ and $\overline{DA}$ are all good, then

$$\text{Area}(\triangle CDE) = \text{Area}(\triangle DEA) = \text{Area}(\triangle EAB) = \text{Area}(\triangle ABC) = \text{Area}(\triangle BCD).$$

It follows that $\text{Area}(\triangle CDE) = \text{Area}(\triangle BCD)$, and hence the fifth diagonal $\overline{EB}$ is also good.

3. Jake tells Jenny that he has three children, two of whom are twins, and that their ages are integers. He also tells her the sum of the ages of his children and the product of their ages. Jenny says that she does not have enough information to determine the ages, but one possibility is that the twins are a prime number of years old. If Jake's twins are teenagers and their age is not prime, find (with proof) the ages of his children.

SOLUTION. Let x, x and y be the ages of Jake's children. Since the twins are teenagers, x is an integer with $13 \le x \le 19$. Furthermore, Let p, p and q be the solution that Jenny discovered with the twins having prime age p. Since Jenny knows the sum of the ages, we must have $2x + y = 2p + q$. Similarly, she is given the product of the ages, so $x^2y = p^2q$.

Multiply the first equation by p^2 and replace p^2q by x^2y to obtain

$$p^2(2x+y) = p^2(2p+q) = 2p^3 + p^2q = 2p^3 + x^2y,$$

and thus $y(x-p)(x+p) = y(x^2-p^2) = 2p^2(x-p)$. Since x is different from p, we can divide by $x-p$ and get $y(x+p) = 2p^2$. In particular, the integer $x+p$ divides $2p^2$. Since p is a prime, the positive divisors of $2p^2$ are 1, p, p^2, 2, $2p$ and $2p^2$. Furthermore, $x+p > 1, 2, p$ and $x+p \neq 2p$ since $x \neq p$. This leaves $x+p = p^2$ or $x+p = 2p^2$ and hence $x = p^2 - p$ or $x = 2p^2 - p$.

Since p is a prime and $13 \leq x \leq 19$, the only possibility here is $p = 3$ and $x = 2p^2 - p = 15$. Using $(x+p)y = 2p^2$, it then follows that $y = 1$. Thus Jake's children have ages 15, 15 and 1. Note that $2x+y = 31$, so $2p+q = 31$ and $q = 25$. As a check, we observe that $(15)^2{\cdot}1 = 3^2{\cdot}25$.

4. Let A and B be points in the plane at distance 2 from each other. Let S be the set of points P such that $(PA)^2 + (PB)^2$ is at most 10. What is the area of S?

SOLUTION. For this problem, it is convenient to use coordinate geometry. Since A and B can be any two points at distance 2 from each other, we can assume that A and B lie on the x-axis, and that A is the point $(-1, 0)$ and B is $(1, 0)$. Now if (x, y) is any point P, then distance $(PA)^2 = (x+1)^2 + y^2$ and $(PB)^2 = (x-1)^2 + y^2$. Then

$$(PA)^2 + (PB)^2 = (x+1)^2 + y^2 + (x-1)^2 + y^2 = 2x^2 + 2y^2 + 2.$$

We want to consider points where this sum is at most 10, so we want $2x^2 + 2y^2 + 2 \leq 10$, or equivalently, $x^2 + y^2 \leq 4$. But $x^2 + y^2$ is the square of the distance from P to the origin, and so we see that the set S consists of all points whose distance from the origin is at most 2. The set S, therefore, is a disk of radius 2 centered at the origin. (By a "disk", we mean a circle, together with its interior.) Using the formula πr^2 for the area of a disk of radius r, we see that S has area 4π.

5. If x, y and z are positive real numbers, show that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}.$$

SOLUTION. If t is a positive real number, then $(t-1)^2 \geq 0$ so $t^2 + 1 \geq 2t$ and, by dividing by t, we have $t + 1/t \geq 2$. In particular, letting $t = (x+y)/(y+z)$, $(y+z)/(z+x)$ and $(z+x)/(x+y)$, in turn, and adding the three inequalities, we obtain

$$\frac{x+y}{y+z} + \frac{y+z}{x+y} + \frac{y+z}{z+x} + \frac{z+x}{y+z} + \frac{z+x}{x+y} + \frac{x+y}{z+x} \geq 6.$$

Note that the two fractions with denominator $x+y$ sum to

$$\frac{y+z}{x+y} + \frac{z+x}{x+y} = \frac{2z}{x+y} + \frac{x+y}{x+y} = \frac{2z}{x+y} + 1$$

and we have similar expressions for the other two pairs. Thus

$$\frac{2z}{x+y} + \frac{2x}{y+z} + \frac{2y}{z+x} + 3 \geq 6$$

and this clearly yields the required inequality.