WISCONSIN MATHEMATICS, SCIENCE AND ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET IV (2005-2006)

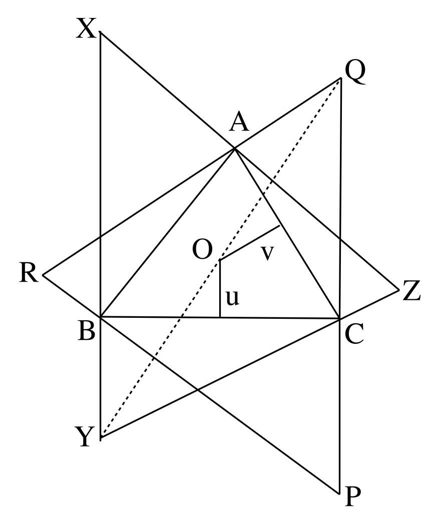
1. Show that for each odd prime number p, there is exactly one positive integer n such that $n(n+p)$ is a perfect square.

SOLUTION. Assume that $n(n+p) = a^2$ for some integer a. Suppose first that n is a multiple of p, so that $n = kp$ for some integer k. Then $n + p = (k+1)p$, and thus $a^2 = n(n+p) = p^2k(k+1)$. Thus p must divide a and $k(k+1) = (a/p)^2$. It follows that the integer a/p is larger than k and smaller than $k+1$, and this is clearly impossible. Thus n is not a multiple of p.

If q is a prime that divides both n and $n+p$, then q divides $(n+p) - n = p$, and thus $q = p$. We have seen, however, that n is not a multiple of p, and it follows that n and $n+p$ have no common prime divisor. Since the product of these two numbers is a square and they have no common prime factor, each of n and $n+p$ must be a square. Write $n+p = u^2$ and $n = v^2$ for positive integers u and v. Then $p = u^2 - v^2 = (u+v)(u-v)$, and since p is prime, this forces $u + v = p$ and $u - v = 1$. Subtracting, we get $2v = p - 1$, so $v = (p-1)/2$ and $n = v^2 = ((p-1)/2)^2$. This is the only possible integer n such that $n(n+p)$ is a square.

Finally, given that p is odd, we see that $n = ((p-1)/2)^2$ is really an integer, and it is a square. Furthermore, $n + p = ((p-1)^2 + 4p)/4 = ((p+1)/2)^2$ is also a square, and so the product $n(n+p)$ is a product of two squares, and hence it is a square.

2. Given $\triangle ABC$, we build two new triangles as follows. First, draw lines through A, B and C perpendicular to $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, respectively. These three lines form one of our new triangles. The other one is also formed by lines through A, B and C, but this time the lines are perpendicular to $\overline{CA}$, $\overline{AB}$ and $\overline{BC}$ respectively. Prove that the two new triangles are congruent.



SOLUTION. Let $\triangle PQR$ and $\triangle XYZ$ be the two new triangles, as shown. Note that $\overline{XY}$ and $\overline{PQ}$ are both perpendicular to $\overline{BC}$, and so they are parallel to each other, and both these lines are parallel to the perpendicular bisector u of side $\overline{BC}$ of the original triangle. Furthermore, u lies exactly halfway between $\overline{XY}$ and $\overline{PQ}$. It follows that line u bisects every line segment joining a point on $\overline{XY}$ to a point on $\overline{PQ}$. Similarly, if we start with side $\overline{CA}$ in place of $\overline{BC}$, we see that $\overline{RQ}$ and $\overline{ZY}$ are parallel, and that every line segment joining a point on one of these lines to a point on the other is bisected by the perpendicular bisector v of $\overline{CA}$.

Now consider line $\overline{YQ}$. By what we have just said, line u goes through the midpoint of $\overline{YQ}$, and line v also goes through this midpoint. It follows that the point where u and v intersect, which is the circumcenter O of $\triangle ABC$, is the midpoint of $\overline{YQ}$. Exactly similar reasoning shows that the circumcenter O is also the midpoint of $\overline{PX}$. Since $\overline{YQ}$ and $\overline{PX}$ are the diagonals of quadrilateral $XQPY$, and these diagonals have a common midpoint, we conclude that quadrilateral $XQPY$ is a parallelogram. Thus $XY = PQ$, and similarly $XZ = PR$ and $YZ = QR$. Therefore $\triangle XYZ \cong \triangle PQR$ by side-side-side.

3. Jake tells Jenny that he has three children, two of whom are twins, and that the ages of all three children are integers. Jake also tells Jenny the sum and the product of the ages of his children. Jenny then says that the age of the non-twin child must be either 9 or 25, but that there is not enough information to determine which of these is correct. Determine (with proof) the product of the ages of Jake's children. [This is a corrected version of the original problem.]

SOLUTION. Let P and S be the product and sum, respectively, of Jake's children's ages. Jenny (who unlike us, knows P and S) has found two possibilities for the ages of the children. We know that in one of her solutions, the age of the non-twin is 9, and we let x be the age of the twins in that solution. (Of course, Jenny knows the value of x, but it is unknown to us.) In Jenny's other solution, the age of the non-twin is 25, and we let y be the age of the twins in that solution. Then $9x^2 = P = 25y^2$ and $9 + 2x = S = 25 + 2y$. We know, therefore, that $9x^2 = 25y^2$, and so by taking square roots (and realizing that ages must be positive) we get $3x = 5y$. Thus $x = 5y/3$, and substituting into the equation $9 + 2x = 25 + 2y$, we get $9 + 10y/3 = 25 + 2y$. Simplifying, we have $4y/3 = 16$, and thus $y = 12$. Also, $x = 5y/3 = 20$. Thus Jenny's two solutions for the ages of Jake's children are 20, 20 and 9, and also 12, 12 and 25. In both cases, the products are the same, as they must be, and we get $P = 3600$. As a check, we observe that the two sums also agree, and $S = 49$.

In the original formulation of this problem, the possible ages for the non-twins were given to be 9 and 16. Using the same methods as above, these numbers yield $x = 14$ and $y = 10\frac{1}{2}$. While the latter is unfortunately not an integer, and we apologize for this, the product and sum are integers. Indeed, $P = 9x^2 = 1764$ and $S = 9 + 2x = 37$.

4. **New Year's Problem.** Prove that for every positive integer m the number $(\sqrt{2006} + \sqrt{2005})^{2m}$ differs from an integer by no more than $1/(4{\cdot}2005)^m$.

SOLUTION. Given a positive integer m, we see that

$$(\sqrt{2006} + \sqrt{2005})^{2m} = (\sqrt{2006} + \sqrt{2005})(\sqrt{2006} + \sqrt{2005}) \cdots (\sqrt{2006} + \sqrt{2005})$$

is a sum of products, where each product has, say, r factors equal to $\sqrt{2006}$ and s factors equal to $\sqrt{2005}$, with $r + s = 2m$. In particular, r and s are either both even or both odd. When they are both even, then the product is certainly an integer. On the other hand, if r and s are both odd, then the product is an integer multiple of $\sqrt{2005}\sqrt{2006}$. Thus $(\sqrt{2006} + \sqrt{2005})^{2m} = A + B\sqrt{2005}\sqrt{2006}$, where A and B are integers that depend on m.

Now, consider $(\sqrt{2006} - \sqrt{2005})^{2m}$. Since each occurrence of $\sqrt{2005}$ is replaced by its negative, there is no effect on terms containing an even number of $\sqrt{2005}$ factors, but terms containing an odd number of such factors are negated. It follows that $(\sqrt{2006} - \sqrt{2005})^{2m} = A - B\sqrt{2005}\sqrt{2006}$, and thus

$$(\sqrt{2006} + \sqrt{2005})^{2m} + (\sqrt{2006} - \sqrt{2005})^{2m} = 2A.$$

This shows that the magnitude of the difference between $(\sqrt{2006} + \sqrt{2005})^{2m}$ and the integer $2A$ is precisely $(\sqrt{2006} - \sqrt{2005})^{2m}$, and it remains to bound the size of the latter number. To this end, note that $(\sqrt{2006} + \sqrt{2005})^2 > (2\sqrt{2005})^2 = 4{\cdot}2005$. Therefore, since $(\sqrt{2006} + \sqrt{2005})(\sqrt{2006} - \sqrt{2005}) = \sqrt{2006}^2 - \sqrt{2005}^2 = 1$, we see that

$$(\sqrt{2006} - \sqrt{2005})^2 = 1/(\sqrt{2006} + \sqrt{2005})^2 \le 1/(4{\cdot}2005),$$

and hence $(\sqrt{2006} - \sqrt{2005})^{2m} \le 1/(4{\cdot}2005)^m$, as wanted. Happy New Year!

5. Let $S = \{2, 3, 22, 23, 32, 33, 222, \ldots\}$ be the set of all positive integers whose decimal digits are 2s and 3s only. Show that no three distinct members of S are in arithmetic progression. (Recall that three integers $a \le b \le c$ are said to be in *arithmetic progression* if $c - b = b - a$.)

SOLUTION. Suppose $a \le b \le c$ belong to S and are in arithmetic progression. Then $c - b = b - a$, and so $a + c = 2b$. Now all of the digits of b lie in the set $\{2, 3\}$, so when computing $2b$, there are no carries, and thus all of the digits of $2b$ lie in the set $\{4, 6\}$. Similarly, when computing $a + c$, there can be no carries, and since $a + c = 2b$, every digit in the sum must be 4 or 6. This implies that a and c have equal numbers of digits since otherwise, the first digit of their sum would be 2 or 3.

Now consider a digit position, say the ith digit from the left. If the ith digits of a and c are different, then one of them is 2 and the other is 3, and so the ith digit of $a + b$ would be 5, which is not the case. It follows that a and c must agree in all their digits, and thus $a = c$. Since $2b = a + c = 2a$, we deduce that $a = b = c$. In other words, three members of S that are in arithmetic progression cannot be distinct.