WISCONSIN MATHEMATICS, SCIENCE AND ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET III (2005-2006)

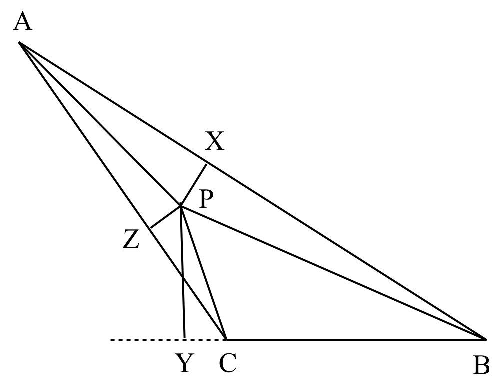
1. Find all real solutions (if there are any) for

$$\frac{w^2+1}{xy} = \frac{x^2+1}{yz} = \frac{y^2+1}{zw} = \frac{z^2+1}{wx} = 2\,.$$

SOLUTION. Observe that none of w, x, y or z can be 0. Also, all of the quantities xy, yz, zw and wx must be positive, and so w, x, y and z must all have the same sign. Given any solution, we can replace all of the unknowns by their negatives to get another solution, so we can assume for now that w, x, y and z are all positive. Using the symmetry of the problem, we can also assume that none of the unknowns is larger than x.

Now $(w^2+1)/xy = (x^2+1)/yz$, so if we cancel y and cross multiply, we get $(w^2+1)z = (x^2+1)x$. But $z \le x$ and $w^2+1 \le x^2+1$, so our equation forces $z = x = w$. Next, $2 = (z^2+1)/wx = (x^2+1)/x^2$, so we deduce that $x^2 = 1$, and hence $x = 1$ because we are assuming that $x > 0$. Thus x, z and w are all equal to 1, and it follows that $y = 1$ too. As we mentioned, we can get one more solution by taking negatives, so the second solution is $w = x = y = z = -1$.

2. Let l and s be the lengths of the longest and shortest altitudes, respectively, of triangle ABC, and let P be a point in the interior of the triangle. Suppose perpendiculars $\overline{PX}$, $\overline{PY}$ and $\overline{PZ}$ are dropped from P to the sides of the triangle. (If the angles of the triangle are not acute, it may be necessary to extend the sides, as shown.) Prove that $s \le PX + PY + PZ \le l$, and that both inequalities are strict unless the triangle is equilateral.



SOLUTION. Let a, b and c be the lengths of the altitudes of $\triangle ABC$ from vertices A, B and C, respectively, and let K denote the area of the triangle. Then $2K = a{\cdot}BC = b{\cdot}CA = c{\cdot}AB$. Now draw lines $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$, as indicated, so that $\triangle ABC$ is split into the three triangles PAB, PBC and PCA. Since the areas of these three triangles sum to K and since $\overline{PX}$, $\overline{PY}$ and $\overline{PZ}$ are their respective altitudes from P, we get $2K = (PX)(AB) + (PY)(BC) + (PZ)(CA)$. Substituting $BC = 2K/a$, $CA = 2K/b$ and $AB = 2K/c$, we obtain, upon dividing by $2K$, the equation $1 = (PX)/c + (PY)/a + (PZ)/b$. Now $l \ge a, b, c \ge s$, so this yields $(PX + PY + PZ)/l \le 1 \le (PX + PY + PZ)/s$, the desired inequalities. Furthermore, if either of these is an equality, then $a = b = c$ and hence $BC = CA = AB$.

3. Find all numbers c such that the equation $2x+3y = c$ has exactly 1000 solutions in positive integers.

SOLUTION. If the equation $2x+3y = c$ has any solutions in positive integers, then obviously c must be a positive integer. We divide the problem into six cases depending upon the remainder when c is divided by 6. The possibilities for c are, of course, $c = 6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$ and $c = 6k+5$ where k is a nonnegative integer. In each case, we compute the number of solutions for a given c.

First, note that if c is even, then y must be even and hence $2 \le y < c/3$. Furthermore, each such y yields precisely one appropriate solution to the equation $2x+3y = c$ since $c-3y$ is a positive even integer and $x = (c-3y)/2$. On the other hand, if c is odd, then y is odd and $1 \le y < c/3$. Again, each such y yields precisely one appropriate solution to the equation $2x + 3y = c$.

Now if $c = 6k$, then y is even and $2 \le y < 2k$. Thus there are $k-1$ possibilities for y in this case. Next, if $c = 6k+1$, then y is odd and $1 \le y \le 2k-1$. Thus there are k possibilities for y and hence k solutions to the equation. If $c = 6k+2$, then y is even, $2 \le y \le 2k$ and there are k possibilities. Similarly, if $c = 6k+3$, then y is odd, $1 \le y \le 2k-1$ and again there are k possibilities for y. If $c = 6k+4$, then y is even, $2 \le y \le 2k$ and there are k possibilities. Finally, if $c = 6k + 5$, then y is odd and clearly $1 \le y \le 2k + 1$. Thus there are $k + 1$ possibilities for y in this case.

It follows that if there are exactly 1000 solutions, then $c = 6{\cdot}1001 = 6006$ or $c = 6{\cdot}1000 + 1 = 6001$ or $c = 6{\cdot}1000 + 2 = 6002$ or $c = 6{\cdot}1000 + 3 = 6003$ or $c = 6{\cdot}1000 + 4 = 6004$ or $c = 6{\cdot}999 + 5 = 5999$. Thus there are precisely six numbers c, namely 5999, 6001, 6002, 6003, 6004 and 6006.

4. Let $n \ge 0$ be an integer and consider numbers of the form $20 \cdots 04$, where there are n zeros between the initial 2 and final 4. Determine which of these numbers can be written as products of four positive integers in arithmetic progression. Recall that a, b, c and d are said to be in arithmetic progression if $b - a = c - b = d - c$.

SOLUTION. The number 24 is the product of 1, 2, 3 and 4, which are four integers in arithmetic progression. We show that if we insert any positive number of 0s between the 2 and 4, then the resulting number is not a product of four integers in arithmetic progression

First, observe that the number $20 \cdots 0$ is a multiple of 8 if there are at least two 0s, and so if we add 4 we get an even number not divisible by 8. We show below that if l is an even integer and $l = abcd$, where a, b, c and d are in arithmetic progression, then l must be a multiple of 8. This will prove that 24 is the only solution to the problem.

Let $k = b - a = c - b = d - c$ and suppose first that k is even. Since l is even, at least one of a, b, c or d must be even, and thus since k is even, all four of a, b, c and d are even and l is a multiple of 16. We can therefore assume that k is odd. If a is even, then b is odd, c is even and d is odd. On the other hand, if a is odd, then b is even, c is odd and d is even. Thus either a and c are both even and differ by $2k$, or else b and d are even and differ by $2k$. Suppose first that a and c are even and write $a = 2m$. Then $c = a + 2k = 2(m + k)$, and so $ac = 4m(m + k)$. But k is odd, so it is clear that m and $m + k$ cannot both be odd. Thus $m(m + k)$ is even, $ac = 4m(m + k)$ is a multiple of 8, and hence $l = abcd$ is also a multiple of 8. This completes the proof in the case where a and c are even, and the proof is the same in the remaining case, where b and d are even.

5. Let S be a string of 2s and 3s such as, for example, 2322. We assign a value $v(S)$ to S by the following process. First, we compute the numbers a_i equal to the sum of the first i digits in the string, starting from the left. Thus in our example, $a_1 = 2$, $a_2 = 2 + 3 = 5$, $a_3 = 2 + 3 + 2 = 7$ and $a_4 = 2 + 3 + 2 + 2 = 9$. The value assigned to the string is the sum of the numbers a_i, so that in our example, $v(S) = 2 + 5 + 7 + 9 = 23$. Find all positive integers that are not values of strings of 2s and 3s.

SOLUTION. Given any string S of 2s and 3s not consisting entirely of 3s, we show how to construct a new string T having the same length and such that $v(T) = 1 + v(S)$. If the right end of S is a 2, just change it to a 3 to create T. This change has no effect on any of the sums a_i except for the last one, which is increased by 1, and so $v(T) = 1 + v(S)$, as wanted. If the right end of S is 3, then since S is not all 3s, there must exist in S at least one 2 followed by a 3. Choose such a pair and create T by interchanging the 2 and 3. If the 2 was in position k, this change increases a_k by 1 and leaves all other a_i unchanged. Again we have $v(T) = 1 + v(S)$.

It is easy to see that a string of all 2s with length n has value $n(n + 1)$ and a string of all 3s with length n has value $3n(n + 1)/2$. The value v of an arbitrary string of length n therefore satisfies $n(n + 1) \le v \le 3n(n + 1)/2$. Indeed, by the observation of the previous paragraph, every integer in this range is the value of some string of length n. For length 1 strings the possible values are 2 and 3; for length 2 the possible values are the integers $6, \ldots, 9$; for length 3 the values are $12, \ldots, 18$; for length 4 the values are $20, \ldots, 30$. For longer strings, we shall see that the ranges overlap, so there are no more missing numbers. The positive integers that are not values are therefore $1, 4, 5, 10, 11$ and 19.

We must determine when the highest possible value for a string of length n is at least as large as the lowest possible value for a string of length $n+1$. The relevant inequality is $3n(n+1)/2 \ge (n+1)(n+2)$. Canceling $n+1$ and simplifying, we get $n \ge 4$, and this shows that our list of missing values is complete.