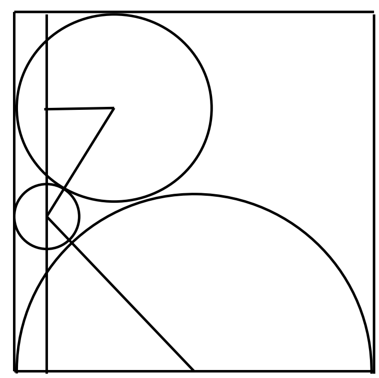
WISCONSIN MATHEMATICS, SCIENCE AND ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET II (2005-2006)

1. Find all real solutions (if there are any) for

$$\frac{w^2+1}{xy} = \frac{x^2+1}{yz} = \frac{y^2+1}{zw} = \frac{z^2+1}{wx} = 1\,.$$

SOLUTION. The equation $(w^2+1)/xy = 1$ yields $xy = w^2+1$, and similarly, $yz = x^2+1$, $zw = y^2+1$ and $wx = z^2+1$. We multiply these four equations and observe that on the left, there are two each of w, x, y and z. This yields $w^2x^2y^2z^2 = (w^2+1)(x^2+1)(y^2+1)(z^2+1)$. But this is impossible because the right side exceeds the left side by at least 1. (To see this, observe that when we expand the parentheses on the right, we get $w^2x^2y^2z^2+1$ plus some additional terms, which are all squares, and hence are nonnegative.) Since our original equations led to a contradiction, we conclude that they have no solution in real numbers.

2. In the previous problem set, we had a square with side length 2, where the lower side of the square was the diameter of a semicircle. A circle tangent to the semicircle and to two sides of the square was drawn, and we saw that its radius was $4-2\sqrt{3}$. Now we add a smaller circle, tangent to the semicircle, the first circle and one side of the square, as shown. Show that the radius of the small circle is exactly $1/3$ the radius of the large circle.



SOLUTION. Draw a vertical line through the center of the small circle and form two right triangles, as shown. The hypotenuse of the upper triangle joins the centers of the two circles and the hypotenuse of the lower triangle joins the center of the small circle and the center of the semicircle. Let R and r be the radii of the large circle and small circle, respectively, and recall that $R = 4-2\sqrt{3}$. Also, of course, the radius of the semicircle is 1.

The length of the hypotenuse of the upper triangle is $R+r$, and its horizontal side has length $R-r$. It follows by the Pythagorean theorem that the square of the length of the vertical side is $(R+r)^2-(R-r)^2 = 4rR$. Similarly, for the lower triangle, the lengths of its hypotenuse and horizontal side are respectively $1+r$ and $1-r$, and hence the square of the length of the vertical side is $4r$. We see from the picture that the total of the lengths of the vertical sides of the two triangles is $2-R$, and this yields the equation $\sqrt{4rR}+\sqrt{4r} = 2-R = 2(\sqrt{3}-1)$, where the last equality follows from the known value of R. Simplification yields $\sqrt{r}(\sqrt{R}+1) = \sqrt{3}-1$, and so $\sqrt{r} = (\sqrt{3}-1)/(\sqrt{R}+1)$. Now observe that $(\sqrt{3}-1)^2 = 4-2\sqrt{3} = R$, and thus since $\sqrt{3}-1 > 0$, we see that $\sqrt{3}-1 = \sqrt{R}$. This yields $\sqrt{r} = \sqrt{R}/\sqrt{3}$, and so $r = R/3$, as wanted.

3. When the UW football team scores, Bucky Badger does a number of pushups equal to UW's total score at that time. For instance, if Wisconsin scores 3 points, then 7 points and then 3 points again, Bucky does first 3, then $3+7=10$ and then $3+7+3=13$ pushups, for a total of $3+10+13=26$. If all of UW's scores are either 3 or 7 points and Bucky does 71 pushups during the game, how many points did UW score in that game?

SOLUTION. If UW scores only four times, then even if all of those were 7 point scores, Bucky would do only $7+14+21+28=70$ pushups, and so we see that UW must have scored at least five times. Suppose now that all of UW's scores were 3 points. If UW scores five times, the number of pushups would be $3+6+9+12+15=45$, and if UW scores six times, the number of pushups would be $45+18=63$. A seventh score would result in more than 71 pushups, and so there were at most six scores.

Five 3-point scores results in 45 pushups and six 3-point scores results in 63 pushups. Since Bucky does 71 pushups in total, it follows that there was at least one 7-point score. But if we change one or more of our original 3-point scores to 7-point scores, that change will add a multiple of 4 to the total number of pushups. Since $71 - 45 = 26$ is not a multiple of 4 and $71 - 63 = 8$ is a multiple of 4, it follows that there must have been six scores, and that at least one of our assumed 3-point scores must be changed to a 7-point score. The change must result in exactly eight additional pushups.

Changing only the sixth (*i.e.* the last) 3 to a 7 adds four pushups. Changing the fifth 3 to a 7 adds four pushups when the fifth score is made and an extra four pushups when the sixth score is made, for a total increase of eight, as needed. If we change one of the earlier 3-point scores to a 7 or change more than one 3 to a 7, that would add more than eight pushups, and that is too many. The only possibility, therefore, is that of the six scores, all but the fifth were 3 points, and the fifth was 7 points. It follows that UW's total score for the game was 22 points.

4. The number of "words" of length n that use only the letters X and Y is exactly 2^n since there are two choices for each of the n positions. Let $F(n)$ be the number of these words that contain two consecutive Xs. For example, if $n = 4$, the words with consecutive Xs are XXXX, XXXY, XXYX, XYXX, YXXX, XXYY, YXXY and YYXX, so $F(4) = 8$. Find $F(10)$ and justify your answer.

SOLUTION. Write $B(n)$ to denote the number of "bad" n-letter words, by which we mean the words that do *not* contain XX. It is easy to see by experiment that $B(1) = 2$, $B(2) = 3$, $B(3) = 5$ and $B(4) = 8$, and so we are tempted to guess that $B(n-2) + B(n-1) = B(n)$ for all $n \geq 3$. To prove that this is correct, observe that if we have any bad word of length $n-1$, we can build a bad word of length n from it by simply appending Y, and that gives all of the length-n bad words that end with Y. To count the length-n bad words that end with X, we note that the last two letters of such a word must be YX. Every such word, therefore, can be constructed from a length-$(n-2)$ bad word by appending YX. It follows that among the $B(n)$ bad words of length n, exactly $B(n-1)$ end in Y and exactly $B(n-2)$ end in X. This proves our conjectured formula $B(n-2) + B(n-1) = B(n)$.

We now compute that $B(5) = 5+8 = 13$, $B(6) = 8+13 = 21$, $B(7) = 13+21 = 34$, $B(8) = 21+34 = 55$, $B(9) = 34+55 = 89$ and $B(10) = 55+89 = 144$. It follows that $F(10) = 2^{10} - B(10) = 1024 - 144 = 880$.

5. Alan and Betty play a game by taking turns removing stones from a pile. The player who takes the last stone loses, and the rules require that at each turn, the number of stones removed must be either 1/3 or 1/4 of the pile, rounded up to the next integer. (For example, if eight stones remain, the number that can be removed is either $8/4 = 2$ or $8/3$ rounded up, which is 3.) If Alan goes first and the pile initially has 100 stones, show that Betty can win regardless of what Alan does.

SOLUTION. On his first turn, Alan can remove either 1/3 of the 100 stones, which is 34 (after being rounded up) or 1/4 of them, which is 25. Betty's strategy at this point is to do the opposite. So if Alan took 1/3, leaving 66, Betty then takes 1/4, which is 17, leaving 49. But if Alan took 1/4, leaving 75, Betty takes 1/3, which is 25, leaving 50. Thus Betty can guarantee that at Alan's second turn, there are 49 or 50 stones left in the pile.

Now on his second turn, if Alan chooses to take 1/3, he takes 17, leaving 32 or 33. In this case, Betty again should make the opposite choice, and she takes 1/4 which is 8 or 9, respectvely, leaving 24. If, on the other hand, Alan takes 1/4 on his second turn, he takes 13 stones, leaving 36 or 37. Then Betty should do the opposite and take 1/3, which is 12 or 13, respectively, leaving 24. This shows that Betty can guarantee that Alan is faced with 24 stones at the start of his third turn.

On his third turn, if Alan takes 1/3, which is 8, he leaves 16 and if he takes 1/4, which is 6, he leaves 18. Again Betty should do the opposite, taking either 1/4 of 16 leaving 12 or 1/3 of 18, also leaving 12. Thus Betty can force Alan to face exactly 12 stones at his fourth turn.

On his fourth turn, Alan takes either 1/3, leaving 8 or 1/4, leaving 9, and again Betty does the opposite, either taking 1/4 from 8 leaving 6 or 1/3 from 9, also leaving 6. Now Alan must take 2 stones, leaving 4 and Betty should take 1/4, leaving 3. Now Alan must take 1, leaving 2, and Betty takes 1 leaving 1, and so Alan loses.