1. Let $Z = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$ be the set of integers, positive, negative and 0. A subgroup $S$ of $Z$ is a nonempty subset with the property that if $x$ and $y$ are members of $S$, then so is $x - y$. Find all subgroups of $Z$ that contain the integer 3.

**SOLUTION.** First, let $S$ be any subgroup of $Z$. Since $S$ is nonempty, choose $s$ in $S$. Then, by definition, $s - s = 0$ is a member of $S$. Next, if $y$ is any member of $S$, then so is $0 - y = -y$, using the fact that 0 is in $S$. Finally, if $x$ and $y$ are members of $S$, then so is $x - (-y) = x + y$. Now suppose that $S$ contains the number 3. Then it contains $3 + 3 = 6$, $3 + 6 = 9$, $3 + 9 = 12$, and so on. In other words, $S$ contains 0 and all positive integer multiples of 3. But if $y$ is in $S$, then so is $-y$. Hence we see that $S$ contains $3Z$, namely all integer multiples of 3. Since it is easy to see that $3Z$ is a subgroup of $Z$ containing 3, this yields one possibility for $S$.

Finally, suppose $S$ is properly larger than $3Z$. Then $S$ contains an integer $z$ not divisible by 3, so if we divide $z$ by 3, we get a remainder $r = 1$ or 2. Say $z = 3q + r$. Since $3q$ is in $S$, we see that $S$ contains $z - 3q = r$. Now $S$ contains all the numbers $3n$ in $3Z$, so $S$ contains all $3n + r$ and $(3n + r) + r = 3n + 2r$. In particular, if $r = 1$, then $S$ contains $3Z$, $3Z + 1$ and $3Z + 2$, so $S = Z$. On the other hand, if $r = 2$, then $S$ contains $3Z, 3Z + 2$ and $3Z + 4$. But it is clear that $3Z + 4 = 3Z + 1$, so again $S = Z$. The possible subgroups are therefore $3Z$ and $Z$ itself.

2. The sides of $\angle APB$ and $\angle CQD$ meet at points $W$, $X$, $Y$ and $Z$, as shown. If the bisectors of these angles are perpendicular, show that the four points $W$, $X$, $Y$ and $Z$ lie on a common circle.

**SOLUTION.** Let the two angle bisectors meet at point $T$ and let $\overline{PT}$ and $\overline{QC}$ meet at point $R$. For convenience, we let $\alpha$ denote the number of degrees in $\angle ZPT = \angle TPY$ and let $\beta$ be the number of degrees in $\angle YQT = \angle TQX$. Since $\angle WZR$ is an exterior angle to $\triangle ZPR$, we have $\angle WZR = \angle ZPR + \angle ZRP = \alpha + \angle ZRP$. Furthermore, $\angle ZRP = \angle TRQ$ and $\triangle RTQ$ is a right triangle, by assumption, so $\angle TRQ + \angle RQT = 90^\circ$. Thus $\angle ZRP = \angle TRQ = 90^\circ - \beta$ and $\angle WZY = \angle WZR = \alpha + \angle ZRP = 90^\circ + \alpha - \beta$. In a similar manner, we can show that $\angle WXY = 90^\circ + \beta - \alpha$. Thus $\angle WZY + \angle WXY = (90^\circ + \alpha - \beta) + (90^\circ + \beta - \alpha) = 180^\circ$ and it follows that quadrilateral $WXYZ$ is inscribed in a circle.

3. Let $N = 100 \ldots 001$ be the integer having $n \geq 0$ zero digits sandwiched between the two ones. If $N$ is a prime number, prove that $n + 1$ is a power of 2.

**SOLUTION.** Note that $N = 100 \ldots 000 + 1 = 10^{n+1} + 1$, since 100 \ldots 000 has $n + 1$ zero digits. We want to show that the positive integer $n + 1$ is a power of 2, and for this it
suffices to show that \( n+1 \) does not have an odd factor. Suppose, by way of contradiction, that \( n+1 = pq \) where \( q \geq 3 \) is odd. Then

\[
N = 10^{pq} + 1 = [10^p + 1] \cdot [(10^p)^q - 1 - (10^p)^{q-2} + (10^p)^{q-3} - \cdots + 1],
\]

and \( N \) is divisible by \( 10^p + 1 \). Since \( 1 < 10^p + 1 < 10^{pq} + 1 = N \), it follows that \( N \) is not a prime number. But \( N \) is given to be a prime, so we cannot have \( n+1 = pq \) with \( q \) odd and \( q \geq 3 \). Therefore \( n+1 \) is indeed a power of 2.

4. Show that no sum of reciprocals of squares of distinct positive integers can ever be as large as 2.

**SOLUTION.** Since \( 1/(1^2) = 1 \), it suffices to show that \( 1/(2^2) + 1/(3^2) + \cdots + 1/(n^2) < 1 \) for all \( n \geq 2 \). For convenience, let us denote the latter sum by \( s_n \). We actually prove the stronger result that \( s_n < 1 - 1/n \) for all \( n \geq 2 \), and we do this by mathematical induction. To start with, when \( n = 2 \), we have \( s_2 = 1/4 < 1/2 = 1 - 1/2 \), so the induction starts properly. Now suppose that \( s_n < 1 - 1/n \) for some integer \( n \geq 2 \). Then

\[
s_{n+1} = s_n + \frac{1}{(n+1)^2} < 1 - \frac{1}{n} + \frac{1}{(n+1)^2} < 1 - \frac{1}{n+1}
\]

since \( 1/n - 1/(n+1) = 1/[n(n+1)] > 1/(n+1)^2 \). Thus \( s_{n+1} < 1 - 1/(n+1) \), as required. We have therefore shown that the inequality \( s_n < 1 - 1/n \) is true for \( n = 2 \) and that, if it is true for \( n \), then it is also true for \( n+1 \). It follows that the inequality holds for all integers \( n \geq 2 \).

5. Prove that there are infinitely many pairs of integers \( x, y \) satisfying the equation \( x^2 - 2y^2 = 1 \).

**SOLUTION.** Suppose \( x \) and \( y \) are positive integers satisfying \( x^2 - 2y^2 = 1 \). Obviously, \( x \) is odd, so \( 2y^2 = x^2 - 1 = (x-1)(x+1) \) is a product of the even integers \( x-1 \) and \( x+1 \). Note that, if \( d \) is an integer dividing both \( x-1 \) and \( x+1 \), then \( d \) divides \( (x+1) - (x-1) = 2 \). Thus \( x-1 \) and \( x+1 \) have only a single factor of 2 in common. Since \( 2y^2 = (x-1)(x+1) \), it therefore follows that one of these factors is the square of an even number and that the other factor is twice a square. Suppose \( x+1 = 2a^2 \) and \( x-1 = (2b)^2 = 4b^2 \). Then \( 2 = (x+1) - (x-1) = 2a^2 - 4b^2 \) and hence \( a^2 - 2b^2 = 1 \). In other words, we have found a smaller integer solution to the given equation. Note that \( 2x = (x+1) + (x-1) = 2a^2 + 4b^2 \), so \( x = a^2 + 2b^2 \). Also \( 2y^2 = (x+1)(x-1) = 8a^2b^2 \), so \( y = 2ab \).

This leads us to guess that if \( a \) and \( b \) are positive integers with \( a^2 - 2b^2 = 1 \) and if \( x = a^2 + 2b^2 \) and \( y = 2ab \), then \( x^2 - 2y^2 = 1 \). This is indeed the case since \( x^2 - 2y^2 = (a^2 + 2b^2)^2 - 8a^2b^2 = (a^2 - 2b^2)^2 = 1 \). We now have a procedure for constructing a bigger solution from any given one. For example, we start with the solution \((3,2), \text{ get } (3^2 + 2 \cdot 2^2, 2 \cdot 3 \cdot 2) = (17, 12) \), and keep going in this manner. Since \( a^2 + 2b^2 > a \) and \( 2ab > b \), we certainly obtain infinitely many solutions in this way.