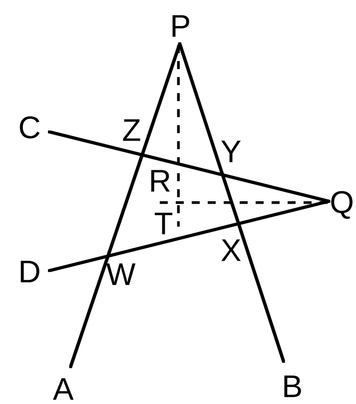
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET V (2004-2005)

1. Let $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ be the set of integers, positive, negative and 0. A *subgroup* S of Z is a nonempty subset with the property that if x and y are members of S, then so is $x - y$. Find all subgroups of Z that contain the integer 3.

SOLUTION. First, let S be any subgroup of Z. Since S is nonempty, choose s in S. Then, by definition, $s - s = 0$ is a member of S. Next, if y is any member of S, then so is $0 - y = -y$, using the fact that 0 is in S. Finally, if x and y are members of S, then so is $x - (-y) = x + y$. Now suppose that S contains the number 3. Then it contains $3 + 3 = 6$, $3 + 6 = 9$, $3 + 9 = 12$, and so on. In other words, S contains 0 and all positive integer multiples of 3. But if y is in S, then so is $-y$. Hence we see that S contains $3Z$, namely all integer multiples of 3. Since it is easy to see that $3Z$ is a subgroup of Z containing 3, this yields one possiblity for S.

Finally, suppose S is properly larger than $3Z$. Then S contains an integer z not divisible by 3, so if we divide z by 3, we get a remainder $r = 1$ or 2. Say $z = 3q + r$. Since $3q$ is in S, we see that S contains $z - 3q = r$. Now S contains all the numbers $3n$ in $3Z$, so S contains all $3n + r$ and $(3n + r) + r = 3n + 2r$. In particular, if $r = 1$, then S contains $3Z$, $3Z + 1$ and $3Z + 2$, so $S = Z$. On the other hand, if $r = 2$, then S contains $3Z$, $3Z + 2$ and $3Z + 4$. But it is clear that $3Z + 4 = 3Z + 1$, so again $S = Z$. The possible subgroups are therefore $3Z$ and Z itself.

2. The sides of $\angle APB$ and $\angle CQD$ meet at points W, X, Y and Z, as shown. If the bisectors of these angles are perpendicular, show that the four points W, X, Y and Z lie on a common circle.



SOLUTION. Let the two angle bisectors meet at point T and let $\overline{PT}$ and $\overline{QC}$ meet at point R. For convenience, we let α denote the number of degrees in $\angle ZPT = \angle TPY$ and let β be the number of degrees in $\angle YQT = \angle TQX$. Since $\angle WZR$ is an exterior angle to $\triangle ZPR$, we have $\angle WZR = \angle ZPR + \angle ZRP = \alpha + \angle ZRP$. Furthermore, $\angle ZRP = \angle TRQ$ and $\triangle RTQ$ is a right triangle, by assumption, so $\angle TRQ + \angle RQT = 90°$. Thus $\angle ZRP = \angle TRQ = 90° - \beta$ and $\angle WZY = \angle WZR = \alpha + \angle ZRP = 90° + \alpha - \beta$. In a similar manner, we can show that $\angle WXY = 90° + \beta - \alpha$. Thus $\angle WZY + \angle WXY = (90° + \alpha - \beta) + (90° + \beta - \alpha) = 180°$ and it follows that quadrilateral $WXYZ$ is inscribed in a circle.

3. Let $N = 100\ldots001$ be the integer having $n \geq 0$ zero digits sandwiched between the two ones. If N is a prime number, prove that $n + 1$ is a power of 2.

SOLUTION. Note that $N = 100\ldots000 + 1 = 10^{n+1} + 1$, since $100\ldots000$ has $n + 1$ zero digits. We want to show that the positive integer $n + 1$ is a power of 2, and for this it suffices to show that $n+1$ does not have an odd factor. Suppose, by way of contradiction, that $n+1 = pq$ where $q \geq 3$ is odd. Then

$$N = 10^{pq} + 1 = [10^p + 1]\cdot[(10^p)^{q-1} - (10^p)^{q-2} + (10^p)^{q-3} - \cdots + 1],$$

and N is divisible by $10^p + 1$. Since $1 < 10^p + 1 < 10^{pq} + 1 = N$, it follows that N is not a prime number. But N is given to be a prime, so we cannot have $n+1 = pq$ with q odd and $q \geq 3$. Therefore $n+1$ is indeed a power of 2.

4. Show that no sum of reciprocals of squares of distinct positive integers can ever be as large as 2.

SOLUTION. Since $1/(1^2) = 1$, it suffices to show that $1/(2^2) + 1/(3^2) + \cdots + 1/(n^2) < 1$ for all $n \geq 2$. For convenience, let us denote the latter sum by s_n. We actually prove the stronger result that $s_n < 1 - 1/n$ for all $n \geq 2$, and we do this by mathematical induction. To start with, when $n = 2$, we have $s_2 = 1/4 < 1/2 = 1 - 1/2$, so the induction starts properly. Now suppose that $s_n < 1 - 1/n$ for some integer $n \geq 2$. Then

$$s_{n+1} = s_n + \frac{1}{(n+1)^2} < 1 - \frac{1}{n} + \frac{1}{(n+1)^2} < 1 - \frac{1}{n+1}$$

since $1/n - 1/(n+1) = 1/[n(n+1)] > 1/(n+1)^2$. Thus $s_{n+1} < 1 - 1/(n+1)$, as required. We have therefore shown that the inequality $s_n < 1 - 1/n$ is true for $n = 2$ and that, if it is true for n, then it is also true for $n+1$. It follows that the inequality holds for all integers $n \geq 2$.

5. Prove that there are infinitely many pairs of integers x, y satisfying the equation $x^2 - 2y^2 = 1$.

SOLUTION. Suppose x and y are positive integers satisfying $x^2 - 2y^2 = 1$. Obviously, x is odd, so $2y^2 = x^2 - 1 = (x-1)(x+1)$ is a product of the even integers $x-1$ and $x+1$. Note that, if d is an integer dividing both $x-1$ and $x+1$, then d divides $(x+1)-(x-1) = 2$. Thus $x-1$ and $x+1$ have only a single factor of 2 in common. Since $2y^2 = (x-1)(x+1)$, it therefore follows that one of these factors is the square of an even number and that the other factor is twice a square. Suppose $x+1 = 2a^2$ and $x-1 = (2b)^2 = 4b^2$. Then $2 = (x+1) - (x-1) = 2a^2 - 4b^2$ and hence $a^2 - 2b^2 = 1$. In other words, we have found a smaller integer solution to the given equation. Note that $2x = (x+1)+(x-1) = 2a^2 + 4b^2$, so $x = a^2 + 2b^2$. Also $2y^2 = (x+1)(x-1) = 8a^2b^2$, so $y = 2ab$.

This leads us to guess that if a and b are positive integers with $a^2 - 2b^2 = 1$ and if $x = a^2 + 2b^2$ and $y = 2ab$, then $x^2 - 2y^2 = 1$. This is indeed the case since $x^2 - 2y^2 = (a^2 + 2b^2)^2 - 8a^2b^2 = (a^2 - 2b^2)^2 = 1$. We now have a procedure for constructing a bigger solution from any given one. For example, we start with the solution $(3, 2)$, get $(3^2 + 2\cdot 2^2, 2\cdot 3\cdot 2) = (17, 12)$, and keep going in this manner. Since $a^2 + 2b^2 > a$ and $2ab > b$, we certainly obtain infinitely many solutions in this way.