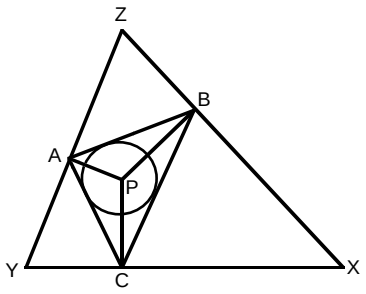
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

PROBLEM SET IV (2004-2005) **JANUARY 2005**

1. I want to buy a pair of fish for my aquarium. The salesman at the pet shop says that if he nets two fish at random from his big tank, the probability that both will be of the same sex is exactly 1/2. (This means that exactly half of all of the possible ways of choosing two fish from the tank result in same-sex pairs.) Assuming that the salesman is telling the truth, prove that the number of fish in the tank is a square.

2. Let P be the center of the inscribed circle of $\triangle ABC$. We then draw $\triangle XYZ$ as shown, with sides passing through points A, B and C and perpendicular to $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$, respectively. Prove that P lies at the intersection of the three altitudes of $\triangle XYZ$.



3. Let x and y be nonnegative real numbers. Prove that

$$4(x^9 + y^9) \geq (x^2 + y^2)(x^3 + y^3)(x^4 + y^4)$$

4. (New Year's Problem) Suppose $\square$ is an operation that defines a new integer $x \square y$ whenever integers x and y are given. Assume that this operation satisfies the following conditions for all nonnegative integers x and y: (a) $1 \square 0 = 1$, (b) $(2x) \square x = 2(x \square x)$ and (c) $(x+1) \square y = (x \square y) + (y^2 + 1) \square 0$. Compute $5 \square 20$.

5. Let us say that a set of three or more prime numbers is *amazing* if the sum of every three of them is also a prime number. For example, the set $\{5, 7, 11, 181\}$ is an amazing set of primes. Prove that no amazing set of four primes can contain 3 and that no amazing set of five primes exists.

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on the problem set page. Remember that solutions usually require a proof or justification.

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DEADLINE: February 3, 2005

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(Please Detach Above)

Last Name	First Name	Grade

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Home Address	Town	Zip Code

Email Address

PROBLEM	SCORE
1	
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PROBLEM SET IV