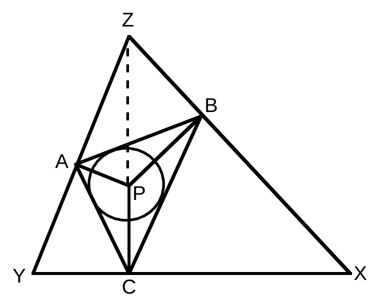
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2004-2005)

1. I want to buy a pair of fish for my aquarium. The salesman at the pet shop says that if he nets two fish at random from his big tank, the probability that both will be of the same sex is exactly 1/2. (This means that exactly half of all of the possible ways of choosing two fish from the tank result in same-sex pairs.) Assuming that the salesman is telling the truth, prove that the number of fish in the tank is a square.

SOLUTION. Given a set of n objects, there are exactly $n(n-1)/2$ subsets of size 2. (There are n ways to choose the first object and $n-1$ ways to choose the second; the division by 2 is because this procedure counts each subset twice.) Now let m and f, respectively, be the numbers of male and female fish in the tank. The total number of pairs of fish, therefore, is $(m+f)(m+f-1)/2$. We know that exactly half of these are same-sex pairs, and thus exactly half consist of one male and one female. But there are m ways to choose the male and f ways to choose the female, yielding mf male-female pairs. We now have $mf = (m+f)(m+f-1)/4$, and thus $4mf = (m+f)^2 - (m+f)$. This yields $m+f = (m+f)^2 - 4mf = (m-f)^2$. The total number of fish in the tank, which is $m+f$, is thus a square.



2. Let P be the center of the inscribed circle of $\triangle ABC$. We then draw $\triangle XYZ$ as shown, with sides passing through points A, B and C and perpendicular to $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$, respectively. Prove that P lies at the intersection of the three altitudes of $\triangle XYZ$.

SOLUTION. Since $\overline{PA}$ bisects $\angle BAC$, we have $\angle PAB = \angle PAC$, and we let α denote the number of degrees in these angles. Similarly, write $\angle PBA = \beta = \angle PBC$ and $\angle PCA = \gamma = \angle PCB$. Then, in $\triangle ABC$, we have $2\alpha + 2\beta + 2\gamma = \angle A + \angle B + \angle C = 180°$, so $\alpha+\beta+\gamma = 90°$. Next, draw line $\overline{PZ}$ and notice that, since $\angle PAZ = 90° = \angle PBZ$, points A and B lie on the circle with diameter $\overline{PZ}$. In this circle, inscribed angles $\angle PAB$ and $\angle PZB$ subtend the same arc, and thus $\angle PZB = \angle PAB = \alpha$. In the same way, we see that $\angle PZA = \beta$, and thus $\angle Z = \alpha + \beta$. Similar reasoning yields $\angle Y = \alpha + \gamma$.

The sum of the angles of a quadrilateral is 360°, so in $YCPZ$ we have $360° = \angle Y + \angle YCP + \angle CPZ + \angle PZY$. Now substitute $\angle Y = \alpha + \gamma$, $\angle YCP = 90°$ and $\angle PZY = \angle PZA = \beta$. Thus since $\alpha + \beta + \gamma = 90°$, we deduce that $\angle CPZ = 180°$. In other words, points C, P and Z lie on a line. This line goes through Z and is perpendicular to $\overline{XY}$, so it is the altitude from Z in $\triangle XYZ$. We know that P lies on this altitude and similarly, P lies on the other two altitudes of $\triangle XYZ$.

3. Let x and y be nonnegative real numbers. Prove that

$$4(x^9 + y^9) \geq (x^2 + y^2)(x^3 + y^3)(x^4 + y^4)$$

SOLUTION. By symmetry, we can clearly suppose that $x \geq y \geq 0$. Now let a and b be any positive integers. Then $x^a \geq y^a$ and $x^b \geq y^b$, so $(x^a - y^a)(x^b - y^b) \geq 0$. This yields $x^{a+b}+y^{a+b} \ge x^a y^b + x^b y^a$ and, by adding $x^{a+b}+y^{a+b}$ to both sides, we obtain $2(x^{a+b}+y^{a+b}) \ge (x^a+y^a)(x^b+y^b)$. When $a=2$ and $b=3$, the above inequality becomes $2(x^5+y^5) \ge (x^2+y^2)(x^3+y^3)$, and when $a=5$ and $b=4$, we have $2(x^9+y^9) \ge (x^5+y^5)(x^4+y^4)$. Consequently

$$4(x^9+y^9) \ge 2(x^5+y^5)(x^4+y^4) \ge (x^2+y^2)(x^3+y^3)(x^4+y^4)$$

as required.

4. (New Year's Problem) Suppose $\square$ is an operation that defines a new integer $x \square y$ whenever integers x and y are given. Assume that this operation satisfies the following conditions for all nonnegative integers x and y: (a) $1\square 0 = 1$, (b) $(2x)\square x = 2(x \square x)$ and (c) $(x+1)\square y = (x \square y) + (y^2+1)\square 0$. Compute $5 \square 20$.

SOLUTION. For convenience, let us write $\bar{y} = (y^2+1)\square 0$. Then setting $x=0$ in condition (c) yields $1\square y = (0\square y)+\bar{y}$ and setting $x=1$ in (c) yields $2\square y = (1\square y)+\bar{y} = (0\square y)+2\bar{y}$. Again, by putting $x=2$ in (c), we have $3\square y = (2\square y)+\bar{y} = (0\square y)+3\bar{y}$ and, by continuing in this manner, it is clear that $x\square y = (0\square y)+x\cdot\bar{y}$ for all nonnegative integers x, y. In particular, if $x=2y$ we have $(2y)\square y = (0\square y)+2(y\cdot\bar{y})$, and if $x=y$ we obtain $2(y\square y) = 2(0\square y)+2(y\cdot\bar{y})$. But $(2y)\square y = 2(y\square y)$ by condition (b), so $0\square y = 2(0\square y)$ and hence $0\square y = 0$ for all integers $y \ge 0$. The formula for $x\square y$ now become $x\square y = x\cdot\bar{y}$. Consequently, we have $\bar{y} = (y^2+1)\square 0 = (y^2+1)\cdot\bar{0}$. But $\bar{0} = (0^2+1)\square 0 = 1\square 0 = 1$ by condition (a), so $\bar{y} = y^2+1$ and hence $x\square y = x\cdot\bar{y} = x(y^2+1)$. Finally, when $x=5$ and $y=20$, we obtain $5\square 20 = 5(20^2+1) = 5(401) = 2005$. Happy New Year!

5. Let us say that a set of three or more prime numbers is *amazing* if the sum of every three of them is also a prime number. For example, the set $\{5, 7, 11, 181\}$ is an amazing set of primes. Prove that no amazing set of four primes can contain 3 and that no amazing set of five primes exists.

SOLUTION. If n is an integer, we can write $n = 3q+r$, where q is the integer quotient when we divide n by 3 and where $r = 0$, 1 or 2 is the remainder. For convenience, let us say that the number n is of type r. In particular, the numbers of type 0 are all divisible by 3. Hence the only prime number of type 0 is 3 itself. Now suppose that S is any amazing set of primes. Then S cannot have three or more members of the same type 1 or 2. Indeed, if p_1, p_2 and p_3 are primes in S of the same type r, then $p_i = 3q_i + r$, so $p_1+p_2+p_3 = 3(q_1+q_2+q_3)+3r$ is divisible by 3. But $p_1+p_2+p_3 > 3$, so this sum is not a prime, contradicting the fact that S is amazing.

Now suppose in addition that S has size ≥ 4. If S contains the prime $p_1 = 3$, then there are at least three other elements of S, each of type 1 or 2. Furthermore, as we saw above, the remaining members cannot all be of the same type. Thus there exist primes p_2 and p_3 in S with $p_2 = 3q_2+1$ and $p_3 = 3q_3+2$. But then $p_1+p_2+p_3 = 3+3(q_2+q_3)+1+2$ is divisible by 3, again a contradiction. In other words, 3 is not a member of S, so the members of S are each of type 1 or 2. But, as we saw, there are at most two members of type 1 and at most two members of type 2. We conclude therefore that the size of S is at most four and that amazing sets of size four cannot contain the prime 3.