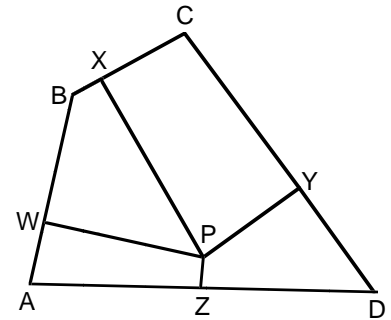


**WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH**  
**PROBLEM SET III (2004-2005)** **DECEMBER 2004**

1. Let  $a, b, c$  and  $d$  be four distinct integers. Find the smallest possible value for  $4(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2$  and prove that your answer is correct.
2. A point  $P$  is chosen inside the quadrilateral  $ABCD$  and perpendiculars  $\overline{PW}$ ,  $\overline{PX}$ ,  $\overline{PY}$  and  $\overline{PZ}$  are drawn from  $P$  to the sides of the quadrilateral, as shown. Suppose  $AW = 2$ ,  $WB = 4$ ,  $BX = 1$ ,  $XC = 3$ ,  $CY = 6$  and  $YD = 4$ . Prove that  $Z$  is the midpoint of  $\overline{DA}$ .
3. Let us say that a set  $S$  of positive integers is *happy* if the smallest member of  $S$  is also the number of members of  $S$ . For example  $\{2, 5\}$  and  $\{3, 5, 9\}$  are happy but  $\{2, 5, 9\}$  is not. For each positive integer  $n$ , let  $H_n$  denote the number of happy subsets of the set  $\{1, 2, 3, \dots, n\}$ . Prove that  $H_n + H_{n+1} = H_{n+2}$  for every positive integer  $n$ .
4. Note that  $(x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$ , and that most of the coefficients of the various powers of  $x$  are even integers. In fact, the only odd coefficients are those of  $x^4$  and of  $1 = x^0$ . Now suppose that  $n$  is any positive integer such that when  $(x + 1)^n$  is expanded in terms of powers of  $x$ , the only odd coefficients in the expansion are the coefficients of  $x^n$  and of 1. Prove that  $n$  must be a power of 2.
5. There are 1001 students at a certain school. Prove that at least one of them must have an even number of friends among the other 1000 students. (You should assume that friendship is symmetric so that if  $A$  is a friend of  $B$ , then  $B$  is a friend of  $A$ .)




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**You are invited to submit a solution even if you get just one problem. Please do not write your solutions on the problem set page. Remember that solutions usually require a proof or justification.**

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RETURN TO:

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DEADLINE:  
 January 3,  
 2005

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 (Please Detach Above)

Last Name	First Name	Grade
School	Town	
Home Address	Town	Zip Code
Email Address		

PROBLEM	SCORE
1	
2	
3	
4	
5	

**PROBLEM SET III**