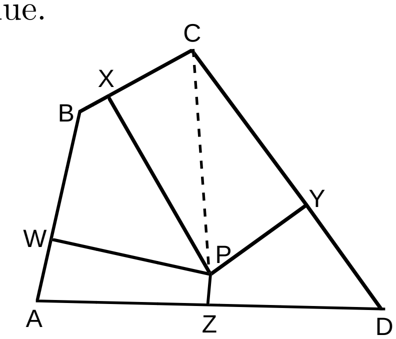
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET III (2004-2005)

1. Let a, b, c and d be four distinct integers. Find the smallest possible value for $4(a^2+b^2+c^2+d^2)-(a+b+c+d)^2$ and prove that your answer is correct.

SOLUTION. Consider $Q=(a-b)^2+(a-c)^2+(a-d)^2+(b-c)^2+(b-d)^2+(c-d)^2$. Each of a, b, c and d occurs in three of the six terms, so if we expand each of the squares and collect like expressions, we have three each of a^2, b^2, c^2 and d^2. We also have the six "cross terms" ab, ac, ad, bc, bd and cd, each occurring with coefficient -2. It follows that $Q=4(a^2+b^2+c^2+d^2)-(a+b+c+d)^2$, and so our task is to minimize the quantity Q subject to the condition that a, b, c and d are distinct integers.

By renaming the variables if necessary, we can assume that $a>b>c>d$. Then each of $(a-b)^2$, $(b-c)^2$ and $(c-d)^2$ is at least $1^2=1$. Also, each of $(a-c)^2$ and $(b-d)^2$ is at least $2^2=4$ and finally, $(a-d)^2$ is at least $3^2=9$. The smallest that Q could possibly be, therefore, is $1+1+1+4+4+9=20$. In fact, we can get $Q=20$ by taking $a=4$, $b=3$, $c=2$ and $d=1$, and thus 20 is the smallest possible value.

2. A point P is chosen inside the quadrilateral $ABCD$ and perpendiculars $\overline{PW}$, $\overline{PX}$, $\overline{PY}$ and $\overline{PZ}$ are drawn from P to the sides of the quadrilateral, as shown. Suppose $AW=2$, $WB=4$, $BX=1$, $XC=3$, $CY=6$ and $YD=4$. Prove that Z is the midpoint of $\overline{DA}$.



SOLUTION. Draw the line $\overline{PC}$. Then $\triangle CPX$ and $\triangle CPY$ are right triangles, so the Pythagorean theorem implies that $(XC)^2+(XP)^2=(PC)^2=(CY)^2+(YP)^2$. Thus $(XC)^2-(CY)^2=(YP)^2-(XP)^2$. Similarly, at the other three vertices of $ABCD$, we get $(YD)^2-(DZ)^2=(ZP)^2-(YP)^2$, $(ZA)^2-(AW)^2=(WP)^2-(ZP)^2$ and $(WB)^2-(BX)^2=(XP)^2-(WP)^2$. We add these four equations and note that the right-hand terms sum to 0. In other words, we have $(XC)^2-(CY)^2+(YD)^2-(DZ)^2+(ZA)^2-(AW)^2+(WB)^2-(BX)^2=0$. Plugging in the known lengths and using $3^2+4^2+4^2=6^2+2^2+1^2$ yields $(ZA)^2-(DZ)^2=0$. Hence $DZ=ZA$ and Z is indeed the midpoint of $\overline{AD}$.

3. Let us say that a set S of positive integers is *happy* if the smallest member of S is also the number of members of S. For example $\{2,5\}$ and $\{3,5,9\}$ are happy but $\{2,5,9\}$ is not. For each positive integer n, let H_n denote the number of happy subsets of the set $\{1,2,3,\ldots,n\}$. Prove that $H_n+H_{n+1}=H_{n+2}$ for every positive integer n.

SOLUTION. The happy subsets of $\{1,\ldots,n+2\}$ are of two types: those that do not contain the number $n+2$ and those that do. The happy subsets of $\{1,\ldots,n+2\}$ that fail to contain $n+2$ are just the happy subsets of $\{1,\ldots,n+1\}$, and so there are exactly H_{n+1} of them. To prove that $H_{n+2}=H_n+H_{n+1}$, therefore, we must show that there are exactly H_n happy subsets of $\{1,\ldots,n+2\}$ that contain $n+2$.

Suppose that X is a happy subset of $\{1,\ldots,n+2\}$ that contains $n+2$. Since the one-element set $\{n+2\}$ is not happy (because $n+2\neq 1$), the set X must contain at least one more member. Also X does not contain 1 since the smallest member of X is the size of X, which is at least 2. Let Y be the set obtained from X by deleting the number $n+2$

and then lowering all other members of X by 1. For example, if $n+2=8$, then X might be the set $\{4,6,7,8\}$, and in that case, $Y=\{3,5,6\}$. Observe that neither $n+2$ nor $n+1$ can lie in Y, and so Y is a subset of $\{1,\ldots,n\}$. Also, Y is happy since it has one fewer member than X and its smallest member is 1 less than the smallest member of X. Since this process is easily reversed, the number of happy subsets of $\{1,\ldots,n+2\}$ that contain $n+2$ equals the total number H_n of happy subsets of $\{1,\ldots,n\}$. Thus $H_{n+2}=H_{n+1}+H_n$.

Note that $H_1=1$ and $H_2=1$. Since each successive number H_n is the sum of the two previous ones, it follows that H_n is the nth Fibonacci number.

4. Note that $(x+1)^4=x^4+4x^3+6x^2+4x+1$, and that most of the coefficients of the various powers of x are even integers. In fact, the only odd coefficients are those of x^4 and of $1=x^0$. Now suppose that n is any positive integer such that the when $(x+1)^n$ is expanded in terms of powers of x, the only odd coefficients in the expansion are the coefficients of x^n and of 1. Prove that n must be a power of 2.

SOLUTION. We proceed by mathematical induction on the exponent n. Since $n=1=2^0$, we can assume that $n\geq 2$ and that the result holds for all exponents m less than n for which $(1+x)^m$ has the appropriate form. Now note that $(1+x)^n=(1+x)(1+x)\cdots(1+x)$ and hence $(1+x)^n$ is equal to the sum of the 2^n products obtained by choosing either 1 or x from each of the n factors $(1+x)$ and then multiplying these terms. For example, if we choose all x's, then the product is x^n, while if we choose all 1's, then the product is $1^n=1$. Of course, the only products that are equal to x are those obtained by choosing one x and the remaining $n-1$ terms equal to 1. Since there are clearly n possible choices for which x to take, we see that $(1+x)^n=x^n+\cdots+nx+1$. But all powers of x other that x^n and $x^0=1$ have even coefficients, so we see that n must be even, say $n=2m$.

Now let us write $(1+x)^m=a_0+a_1x+\cdots+a_{m-1}x^{m-1}+a_mx^m$, where $a_0=a_m=1$. Then $(1+x)^n=[(1+x)^m]^2$ is the square of the above expression. Thus $(1+x)^n$ is the sum of the squares $(a_0)^2,(a_1x)^2,(a_2x^2)^2,\ldots,(a_mx^m)^2$, along with all the "cross terms" $2(a_ix^i)(a_jx^j)$ with $i\neq j$. But the latter cross terms all have even coefficients and hence do not affect the evenness or oddness of the coefficients in the sum. It follows that $(a_1)^2,(a_2)^2,\ldots,(a_{m-1})^2$ are all even, and hence $a_1,a_2,\ldots,a_{m-1}$ are all even. Thus the coefficients of $(1+x)^m$ have the same properties as those of $(1+x)^n$. Since $m<n$, we conclude that $m=2^t$ is a power of 2 and therefore $n=2m=2^{t+1}$ is also a power of 2.

5. There are 1001 students at a certain school. Prove that at least one of them must have an even number of friends among the other 1000 students. (You should assume that friendship is symmetric so that if A is a friend of B, then B is a friend of A.)

SOLUTION. Imagine that during one day, every pair of friends shakes hands exactly once. Since each handshake results in two hands being shaken, we see that if we count the total number of times during the day that a hand was shaken, we obtain an even number. There are 1001 right hands among the students at the school, and since 1001 is an odd number, it cannot be true that each of these hands was shaken an odd number of times. (This is because a sum of an odd number of odd numbers must be odd, but the total number of shaken hands is even.) It follows that at least one hand was shaken an even number of times, and thus the owner of that hand has an even number of school friends.