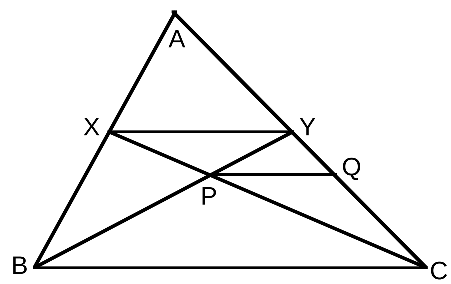
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2004-2005)

1. Do there exist primes p and q such that the quadratic equation $px^2 - qx + p = 0$ has a rational (i.e. a ratio of integers) solution? If so, find all possibilities.

SOLUTION. It is clear that if x is a real root of $px^2 - qx + p = 0$, then $x > 0$. Now let $x = m/n$, where m and n are positive integers. We can clearly assume that this fraction is in lowest terms, so that m and n have no nontrivial factor in common. Plugging $x = m/n$ into the quadratic equation and multiplying by n^2 yields $pm^2 - qmn + pn^2 = 0$ and hence $m(qn - pm) = pn^2$. Thus m divides pn^2 and, since m and n have no proper factors in common, we see that m divides p. In other words, $m = 1$ or p, and similarly $n = 1$ or p. Thus $x = m/n = 1$, p or $1/p$. If $x = 1$, then the quadratic equation yields $q = 2p$, contradicting the fact that q is prime. On the other hand, if $x = p$ or $1/p$, then the equation yields $q = p^2 + 1$. Here, if p is odd, then q is an even number ≥ 10, again contradicting the fact that q is prime. Thus p must be even, so $p = 2$, $q = 5$ and the roots of $2x^2 - 5x + 2 = 0$ are 2 and $1/2$.

2. In $\triangle ABC$, line $\overline{XY}$ is drawn parallel to $\overline{BC}$, where X lies on side $\overline{AB}$ and Y lies on side $\overline{AC}$, as shown. Let P be the point where $\overline{BY}$ meets $\overline{CX}$, and suppose that the line through P parallel to $\overline{BC}$ meets $\overline{YC}$ at Q. If $CQ = 2$ and $QY = 1$, find length AC and prove that your answer is correct.



SOLUTION. Since $\overline{PQ}$ is parallel to $\overline{BC}$, we see that $\triangle YPQ$ is similar to $\triangle YBC$. Thus $YP/YB = YQ/YC = 1/3$, and in particular, $YP/PB = 1/2$. Again, since $\overline{XY}$ is parallel to $\overline{BC}$, we see that $\triangle PYX$ is similar to $\triangle PBC$, and hence $XY/BC = PY/PB = 1/2$. Finally, since $\overline{XY}$ is parallel to $\overline{BC}$, we see that $\triangle AXY$ is similar to $\triangle ABC$. Hence $AY/AC = XY/BC = 1/2$, and it follows that $AY = YC = 3$. Thus $AC = AY + YC = 6$.

3. Given a real number x, the *floor* of x, denoted $\lfloor x \rfloor$, is defined to be the largest integer n such that $n \leq x$. Similarly, the *ceiling* of x, denoted $\lceil x \rceil$, is the smallest integer m such that $x \leq m$. We also define $\mathrm{av}(x)$ to be the average of $\lfloor x \rfloor$ and $\lceil x \rceil$. Prove that $\lfloor x + y \rfloor \leq \mathrm{av}(x) + \mathrm{av}(y) \leq \lceil x + y \rceil$ for all real numbers x and y.

SOLUTION. If x is an integer, then $\lfloor x + y \rfloor = x + \lfloor y \rfloor$ and $\lceil x + y \rceil = x + \lceil y \rceil$. Also $x = \lfloor x \rfloor = \lceil x \rceil = \mathrm{av}(x)$. Since $\lfloor y \rfloor \leq \mathrm{av}(y) \leq \lceil y \rceil$, adding $x = \mathrm{av}(x)$ to all these terms yields $\lfloor x + y \rfloor = x + \lfloor y \rfloor \leq \mathrm{av}(x) + \mathrm{av}(y) \leq x + \lceil y \rceil = \lceil x + y \rceil$, as wanted. Similarly, the result holds if y is an integer.

Finally, suppose that neither x nor y is an integer. Then for some integers a and b, we have $a < x < a + 1$ and $b < y < b + 1$. Then $\lfloor x \rfloor = a$, $\lceil x \rceil = a + 1$, and hence $\mathrm{av}(x) = a + 1/2$. Similarly, $\mathrm{av}(b) = b + 1/2$, so $\mathrm{av}(x) + \mathrm{av}(y) = a + b + 1$. Now $a + b < x + y$, so $\lceil x + y \rceil \geq a + b + 1 = \mathrm{av}(x) + \mathrm{av}(y)$. Also $x + y < a + b + 2$, so $\lfloor x + y \rfloor \leq a + b + 1 = \mathrm{av}(x) + \mathrm{av}(y)$. This completes the proof.

4. Recall that the *Fibonacci numbers* are the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, where after the initial two 1's, each number in the sequence is the sum of the previous two. Prove that there is no positive integer m such that the sum of every m consecutive Fibonacci numbers is odd. Also, determine all positive integers n such that the sum of every n consecutive Fibonacci numbers is even.

SOLUTION. Let us denote the terms of the Fibonacci sequence by $f_1, f_2, f_3, \ldots$, so that $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, and so on. Since $f_{k+1} = f_k + f_{k-1}$ for all $k \geq 2$, we see that $f_{k-1} + f_k + f_{k+1} = 2f_{k+1}$ is even. In other words, the sum of every three consecutive Fibonacci numbers is even. It follows that if the sum of every n consecutive Fibonacci numbers is even, then the same is true of every $n+3$ consecutive numbers and also of every $n-3$ consecutive numbers when $n > 3$. Similarly, if the sum of every m consecutive Fibonacci numbers is odd, then the same is true of every $m+3$ consecutive numbers and also of every $m-3$ consecutive numbers when $m > 3$.

Suppose now that the sum of every m consecutive Fibonacci numbers is odd. Then by continually subtracting 3 from m, we can assume that $m = 1$, 2 or 3. But these possibilities do not hold since f_3, $f_1 + f_2$ and $f_1 + f_2 + f_3$ are all even. Conversely, suppose that the sum of every n consecutive Fibonacci numbers is even. Again, by continually subtracting 3 from n, we can assume that $n = 1$, 2 or 3. Now $n = 1$ and $n = 2$ fail, since f_1 and $f_2 + f_3$ are odd. On the other hand, we know that the result is true for $n = 3$, and by continually adding 3 to n, we see that if n is a multiple of 3, then the sum of any n consecutive Fibonacci numbers is even.

5. Let A, B and C be subsets of a finite set S and assume that every element of S is in at least one of A, B or C. Prove that there is a subset X of the set $\{A, B, C\}$ such that at least $4/7$ of the elements of S are in an odd number of the members of X.

SOLUTION. Let a be the number of elements of S that are in A but not in B or C. Similarly, let b be the number of elements that are only in B, and let c be the number of elements that are only in C. Also, let u be the number of elements in both A and B but not in C, and similarly, let v be the number of elements in B and C but not A, and let w be the number of elements in A and C but not B. Finally, let z be the number of elements that are in all three sets. We have now accounted for all of the elements of S, and we let n be the total number of these elements, so that $a+b+c+u+v+w+z = n$.

For each of subset X of $\{A, B, C\}$, write $o(X)$ to denote the number of elements of S that occur in an odd number of members of X. For example, if X consists just of A, then $o(X)$ is simply the number of elements in A, which is $a + u + w + z$; if $X = \{A, B\}$, then $o(X) = a+b+v+w$; and if $X = \{A, B, C\}$, then $o(X) = a+b+c+z$. Excluding the empty set, there are a total of seven possibilities for X, and we want to compute the average of the seven corresponding numbers $o(X)$. For each of the possibilities for X, we find that $o(X)$ is the sum of four of the seven quantities a, b, c, u, v, w and z. Furthermore, each of these seven quantities occurs in four of the seven sums. It follows that the sum of the seven numbers $o(X)$ is $4(a + b + c + u + v + w + z) = 4n$, and thus the average value of $o(X)$ is $4n/7$. Finally, we observe that it is impossible that all values of $o(X)$ are below average, and thus for some choice of X, we must have $o(X) \geq 4n/7$, as wanted.