WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

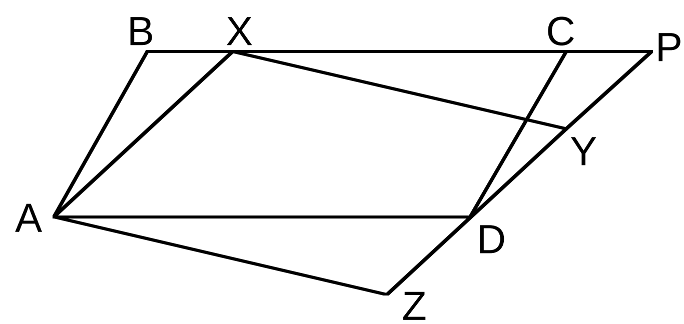
SOLUTIONS TO PROBLEM SET I (2004-2005)

1. Find all positive integers c such that it is possible to write $c = \frac{a}{b} + \frac{b}{a}$ with positive integers a and b.

SOLUTION. We can suppose that there is no prime number that divides both a and b since otherwise both fractions a/b and b/a can be reduced without affecting c.

Suppose that $a > 1$ and choose some prime divisor p of a. Multiply both sides of the given equation by ab to get $abc = a^2 + b^2$. Then $b^2 = abc - a^2 = a(bc - a)$ is a multiple of a, and since a is a multiple of p, it follows that b^2 is a multiple of p. But we know that no prime divides both a and b, so b is not a multiple of p. Thus b^2 cannot be a multiple of p, and this is a contradiction. We obtained this contradiction by assuming $a > 1$, and so we conclude that $a = 1$. Similarly $b = 1$, and it follows that $c = 2$ is the only possibility.

2. In the figure, $ABCD$ and $AXYZ$ are parallelograms such that point X lies on side $\overline{BC}$ and point D lies on side $\overline{YZ}$. Prove that the two parallelograms have equal areas.



SOLUTION. Extend $\overline{BC}$ and $\overline{ZY}$ to meet at the point P, as shown. (Note that these lines must really meet.) Now $AXPD$ is a parallelogram since $\overline{XP}$ is the same line as $\overline{BC}$, and hence it is parallel to $\overline{AD}$, while $\overline{DP}$ is the same line as $\overline{ZY}$, and hence it is parallel to $\overline{AX}$.

Parallelograms $ABCD$ and $AXPD$ share "base" AD and have the same "height", which is the perpendicular distance between $\overline{AD}$ and $\overline{BP}$. Thus $ABCD$ and $AXPD$ have equal areas. Similarly, $AXYZ$ and $AXPD$ have equal areas since they share the same "base" $\overline{AX}$ and have the same "height", which is the perpendicular distance from $\overline{AX}$ to $\overline{ZP}$. Thus each of the two given parallelograms has area equal to that of parallelogram $AXPD$, and consequently the two given parallelograms have equal areas.

3. Find all prime numbers p such that $p^2 = n^3 + 1$ for some integer n.

SOLUTION. If $n = 0$, then $p^2 = n^3 + 1 = 1$, so $p = 1$ and this is not true since 1 is not a prime. If $n < 0$, then $p^2 = n^3 + 1 \leq 0$, again a contradiction. Thus we know that $n > 0$. Now $p^2 = n^3 + 1 = (n + 1)(n^2 - n + 1)$, and since p is prime, there are just three possibilities, which we investigate in turn. Either each factor on the right equals p, or $n + 1 = 1$ or $n^2 - n + 1 = 1$.

First, suppose that $n + 1 = p = n^2 - n + 1$. Then $n^2 - 2n = 0$, so $n \neq 0$ implies that $n = 2$. In this case, we get $p^2 = 9$, and thus $p = 3$ is a solution. Next, if $n + 1 = 1$, we get $n = 0$, and we have already eliminated this possibility. Finally, if $n^2 - n + 1 = 1$, we have $n^2 - n = 0$, so $n = 1$. Thus $p^2 = 2$ and p is not even an integer. We have therefore found only one possibility, namely $p = 3$.

4. Find all real numbers a with the property that the polynomial equation $x^{10}+ax+1=0$ has a real solution r such that $1/r$ is also a solution.

SOLUTION. Since r and $1/r$ are solutions of the given polynomial equation, we have $r^{10}+ar+1=0$ and $(1/r)^{10}+a(1/r)+1=0$. If $a=0$, then $r^{10}+1=0$, and this is a contradiction since r is a real number and 10 is even. Thus $a \neq 0$. Next, we simplify the second equation by multiplying both sides by r^{10}. We get $1+ar^9+r^{10}=0$, and therefore $ar=-(1+r^{10})=ar^9$, where the first equality comes from the first equation above.

Since $a \neq 0$, we can cancel a in the equation $ar=ar^9$ to deduce that $r=r^9$. But $r \neq 0$ since $1/r$ is a number, and so we can cancel r to get $r^8=1$. It is easy to see that this equation has just two real solutions, namely $r=1$ and $r=-1$. Finally, plugging these into the equation $r^{10}+ar+1=0$ we obtain precisely two possibilities for a. These are $a=2$ and $a=-2$.

5. Several people started with \$300 each, and played a game with the following strange rules. *Each player pays \$10 to the house at the beginning of each round. During each round, one active player is declared the loser, and he distributes all of his money in equal amounts to the remaining players. The loser must then leave, but all of the other players go on to the next round. The game is over when only one player remains.* At the end of the game, the surviving player was surprised to discover that he had exactly \$300, equaling his starting amount. How many players were there at the beginning?

SOLUTION. Suppose there were n players at the beginning of the game, and let us keep track of the total amount of money in the possession of all active players. At the start of the game, there are n players, each with \$300, so the total number of dollars is $T_0=300n$. At the start of the first round, each of the n players gives \$10 to the house, so the total is reduced by $10n$ to $T_1=300n-10n$. At the end of this round, one player must leave, but the total does not change since he gives all of his money to the remaining players. Next, at the start of the second round, each of the remaining $n-1$ players gives \$10 to the house, so the total is reduced by $10(n-1)$ to $T_2=300n-10n-10(n-1)$. Similarly, at the start of the third round we have $n-2$ players, so the total is reduced to $T_3=300n-10n-10(n-1)-10(n-2)$.

The game ends after $n-1$ rounds, since at that point only one player is left. It follows that the total amount of his funds is $T_{n-1}=300n-10[n+(n-1)+(n-2)+\cdots+2]$, and by summing the arithmetic series we have $T_{n-1}=300n-5(n+2)(n-1)$. By assumption, this player ends with \$300, so $300=T_{n-1}=300n-5(n+2)(n-1)$ and therefore $300(n-1)=5(n+2)(n-1)$. Now we were told that several people started the game, so this means that $n>1$ and we can cancel the nonzero factor $n-1$ to conclude that $n=58$. Our assumption that the loser distributes his funds equally to the remaining players is used to guarantee that, at each round, all active players have the required \$10 to pay the house. If we allowed players to borrow money, or equivalently to have negative funds, then the equal distribution of funds assumption is no longer needed.