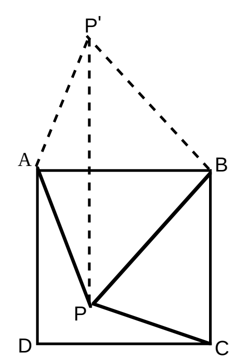
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET V (2003-2004)

1. Ten people attend a party at which some of the guests shake hands with other guests and no two people shake hands with each other more than once. If there are a total of 26 handshakes, prove that there must be three people at the party each of whom has shaken hands with both of the others.

SOLUTION. There were 26 handshakes and two hands in each shake, so there were 52 instances of a hand shaking another hand. Thus, on average, each person shook 5.2 hands, and since not everyone can be below average, some person (say Alice) must have shaken at least six other peoples' hands. Let X be the set of people whose hands Alice shook and let Y be the set of those (other than Alice, herself) whose hands she did not shake. Let $k \geq 6$ be the number of people in X and observe that Y contains exactly $9-k$ people.

Suppose no three people all shook hands with each other. Then no two people in X shook hands, and so the people in X shook hands only with Alice and perhaps with people in Y. In addition to the k handshakes involving Alice, therefore, there were at most $k(9-k)$ other handshakes involving people in X, and this accounts for at most $k+k(9-k)$ handshakes. Since $6 \leq k \leq 9$, we have $k+k(9-k) \leq 24 < 26$, and so some people in set Y must have shaken hands with each other. Say Bill and Carol are members of Y who shook hands. Then no one shook hands with both Bill and Carol, and thus each person in X shook at most $(9-k)-1 = 8-k$ hands in Y. Also, there were at most two handshakes among the at most three people in Y, and so the total number of handshakes was at most $k+k(8-k)+2$. This is less than 26 for all possible values of k, and we have a contradiction.

2. In the diagram, $ABCD$ is a square and P is a point inside it. Show that the distances PA, PB and PC satisfy the inequality $PA+PC \geq \sqrt{2}\cdot PB$. (Actually, it is irrelevant that P is *inside* the square. The inequality is valid for all points P in the plane.)



SOLUTION. Rotate $\triangle BPC$ clockwise 90° about point B so that point B stays fixed, C moves to A and P moves to P', as shown. Of course, $\triangle BPC \cong \triangle BP'A$, and thus $P'A = PC$ and $PA+PC = PA+P'A$. But $PA+P'A$ is the length of the "broken line" from P to A to P', and thus $PA+PC = PA+P'A \geq PP'$. Now look at $\triangle PBP'$, and observe that this is an isosoceles right triangle with side BP. We conclude that $PP' = \sqrt{2}\cdot BP$, and we are done.

3. Suppose that p and q are prime numbers and that m is an integer. If $n = \dfrac{p}{q}+\dfrac{q}{p}-\dfrac{m^2}{pq}$ is a positive integer, find all possibilities for n.

SOLUTION. We get $n=1$ if $p=q=m$ and $n=2$ if $p=q$ and $m=0$. To show that there are no other possibilities for the positive integer n, it suffices to prove that in all cases, $n \leq 2$. If $p=q$, then $n = 2-m^2/p^2$ and $n \leq 2$, as wanted, and so we can assume that $p \neq q$. By symmetry, we can further assume that $p<q$. Since $n = \dfrac{p^2+q^2-m^2}{pq}$ is an integer, we see that $p^2+q^2-m^2$ is a multiple of q, and thus q divides $p^2-m^2 = (p-m)(p+m)$. But since q is prime, it must divide one of the factors $p-m$ or $p+m$.

We can assume that $m \geq 0$. If q divides $p+m$, then since $p+m > 0$, we have $q \leq p+m$, and so $q - p \leq m$. This yields $(q-p)^2 \leq m^2$, and thus $p^2 + q^2 - m^2 \leq 2pq$ and we have $n \leq 2$, as wanted. (Note that we get $n = 2$ if $m = q - p$.) Now suppose that q divides $p - m$. Since $p - m \leq p < q$, the divisibility forces $p - m \leq 0$. If $p - m = 0$, then $p = m$ and $n = q^2/pq = q/p$, which is not an integer. The remaining possibility is $p - m < 0$, and since $p - m$ is a multiple of q, it follows that $p - m \leq -q$. Then $p + q \leq m$, and so $(p+q)^2 \leq m^2$. We deduce that $p^2 + q^2 - m^2 \leq -2pq$, and this yields $n \leq -2$, and we are done.

4. Let f be a function such that $f(n)$ is an integer for each integer n. Prove that there is some integer n such that $f(f(n)) \neq n + 3$.

SOLUTION. Suppose that $f(f(n)) = n+3$ for all integers n. Then $f(n+3) = f(f(f(n))) = f(n) + 3$ for all n. Now define $g(n) = f(n) - n$, so that $f(n) = n + g(n)$. Then $g(n+3) = f(n+3) - (n+3) = f(n) - n = g(n)$, and thus the function g has equal values on any two numbers that differ by 3 or a multiple of 3. For every integer k, therefore, we have $g(3k) = g(0)$, and similarly, $g(3k+1) = g(1)$ and $g(3k+2) = g(2)$, and thus $g(0)$, $g(1)$ and $g(2)$ are the only values that the function g can take. For simplicity of notation, let us write $g(0) = a$, $g(1) = b$ and $g(2) = c$. Then f adds a to every number of the form $3k$; it adds b to numbers of the form $3k+1$ and it adds c to numbers of the form $3k+2$.

Now $f(0) = a$ and if a has the form $3k$, then $3 = f(f(0)) = f(a) = a + a$, and so $2a = 3$, which is impossible. Similarly, $f(1) = b + 1$, so if b has the form $3k$, then $b+1$ is of the form $3k+1$ and $4 = f(f(1)) = f(b+1) = b + 1 + b$, and we get $2b = 3$, which is impossible. We get a similar contradiction if c has the form $3k$.

Now suppose that a has the form $3k+1$. Then $3 = f(f(0)) = a + b$, and so b has the form $3k+2$. If c has the form $3k+1$, then $f(2) = 2 + c$ has the form $3k$, and so $5 = f(f(2)) = 2 + c + a$ and $c + a = 3$. But this is false since both a and c are of the form $3k+1$, so c must have the form $3k+2$. Then $f(2) = 2 + c$ has the form $3k+1$ and $5 = f(f(2) = 2 + c + b$. We deduce that $c + b = 3$, which is wrong since b and c both have the form $3k+2$. We have now derived a contradiction from the assumption that a has the form $3k+1$. Similar reasoning yields a contradiction in the remaining case, where a has the form $3k+2$, and this completes the proof.

5. All of an 8×8 chessboard, with the exception of one square, is covered by 1×3 rectangular tiles. How many of the 64 squares can occur as the uncovered square?

0	1	2	0	1	2	0	1
1	2	0	1	2	0	1	2
2	0	1	2	0	1	2	0
0	1	2	0	1	2	0	1
1	2	0	1	2	0	1	2
2	0	1	2	0	1	2	0
0	1	2	0	1	2	0	1
1	2	0	1	2	0	1	2

SOLUTION. Write a number in each box of the chessboard as shown, and count that there are 21 zeros, 22 ones and 21 twos. Since, each of the 3×1 tiles must cover one each of 0, 1 and 2, the uncovered box must be labeled 1. But we could also have numbered the boxes in mirror-image fashion, with the "stripes" running downhill as one moves from left to right. In this numbering too, the uncovered box must be labeled 1, so we look for 1s in the diagram whose mirror image position is also labeled 1. There are exactly four such: two in the third row and two in the sixth. These four boxes are the only ones that can possibly occur as the uncovered box. A little trial and error shows that box 3 in row 3 really can occur, and then by symmetry we see that the other three boxes we found can also occur. The answer to the problem, therefore, is 4.