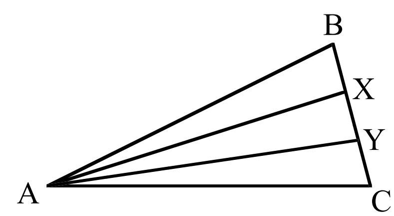
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2003-2004)

1. (New Year's Problem) Consider the following numbers, each of which is slightly larger than 5/3: $a_0 = 1.7$, $a_1 = 1.67$, $a_2 = 1.667$ and in general, $a_m = 1.66\cdots67$, where there are exactly m sixes in a_m. Find the integer of the form na_m that is closest to 2000, where n and m are integers. Prove that your answer is correct.

SOLUTION. For each integer $m \geq 0$, let us write $b_m = 166\cdots67 = 10^{m+1}a_m$ so that b_m is an integer not divisible by 2 or 5. If k is an integer of the form $k = na_m$, where n and m are integers, then $10^{m+1}k = n{\cdot}10^{m+1}a_m = nb_m$. Thus b_m divides $10^{m+1}k$ and, since b_m is not divisible by 2 or 5, we see that b_m divides k. If $m = 0$, then $b_m = b_0 = 17$, so k is a multiple of 17, and the closest multiple of 17 to 2000 is $17{\cdot}118 = 2006$. Next, if $m = 1$, then $b_m = b_1 = 167$, and the closest multiple of 167 to 2000 is $167{\cdot}12 = 2004$. If $m = 2$, then $b_m = b_2 = 1667$, and $1667{\cdot}1$ is its closest multiple to 2000. Finally, if $m \geq 3$, then $b_m \geq b_3 = 16667$ so the multiples of b_m are either 0 or significantly larger than 2000. Thus, the answer is $a_1{\cdot}10^2{\cdot}12 = b_1{\cdot}12 = 2004$. Happy New Year!

2. Let $\overline{AX}$ and $\overline{AY}$ be the trisectors of $\angle A$ in $\triangle ABC$, as shown. Prove that it is *not* true that $BX = XY = YC$.



SOLUTION. Point X lies on the bisector of $\angle BAY$, and so X is equidistant from $\overline{AB}$ and $\overline{AY}$. Writing h to denote the perpendicular distance from X to each of these lines, we see that the area of $\triangle BAX$ is equal to $(h/2)AB$, and similarly, the area of $\triangle YAX$ is $(h/2)AY$. If X is the midpoint of $\overline{BY}$, then the areas of $\triangle BAX$ and $\triangle YAX$ are equal, and thus $(h/2)AB = (h/2)AY$, and we deduce that $AB = AY$. Then $\triangle BAY$ is isosceles, and thus the angle bisector $\overline{AX}$ is also an altitude. This proves that if $BX = XY$, then $\overline{AX}$ is perpendicular to $\overline{BC}$. Similarly, if $XY = YC$, then $\overline{AY}$ is perpendicular to $\overline{BC}$. But there can be only one line from A that is perpendicular to $\overline{BC}$, and so it is not possible to have both $BX = XY$ and also $XY = YC$.

3. Suppose that A and B are nonempty subsets of the set of positive integers, and assume that each positive integer is in one of these two sets, but that no integer lies in both. Suppose also that each of A and B is closed under triple sums. In other words, if x, y and z are any three (not necessarily different) members of A, then $x + y + z$ lies in A, and similarly for B. Find all possibilities for the sets A and B.

SOLUTION. The set consisting of all odd positive integers is closed under triple sums, and so is the set of all even positive integers. We show that these are the only possibilities for A and B.

Since every positive integer is in one of the two sets, the number 1 must be somewhere, and we can suppose that 1 is in A. (Otherwise, interchange the roles of A and B.) Since A is closed under triple sums, we know that the number $3 = 1 + 1 + 1$ must lie in A, and then $5 = 3 + 1 + 1$ also lies in A. Then $7 = 5 + 1 + 1$ lies in A, and continuing like this, we see that A contains all the odd numbers.

Assume that A also contains some even number n. The next even number is $n+2 = n+1+1$, which must also lie in A because of the triple-sum closure property. Similarly the following even number lies in A, and continuing like this, we see that A contains every even number that is at least n. Since A and B have no members in common, we conclude that B contains no odd numbers at all, and no even numbers as big as n. This leaves only finitely many numbers that can be in B, and since B is not empty, it must have a largest member k. But then $3k = k+k+k$ lies in B, and this is a contradiction since $3k$ exceeds k. Therefore, the assumption that A contains some even number must be wrong, and thus A is exactly the set of odd positive integers, and it follows that B is exactly the set of even positive integers.

4. As in Problem 1 of the previous problem set, let a and b be positive numbers with $(1/a) + (1/b) = 1$, and let $[x]$ denote the largest integer that does not exceed the number x. If neither a nor b can be written as the ratio of two integers, show that each positive integer is either of the form $[ma]$ or $[mb]$ for some positive integer m.

SOLUTION. Let t be a positive integer and suppose, by way of contradiction, that $t \neq [ma]$ and $t \neq [mb]$ for any integer m. If ra is the largest integer multiple of a at most equal to t, then $ra \leq t$ and $(r+1)a > t$. In fact, $(r+1)a \geq t+1$ since otherwise we would have $t < (r+1)a < t+1$ and $t = [(r+1)a]$. Furthermore, $ra \neq t$ and $(r+1)a \neq t+1$ since a is not the ratio of two integers. In other words, we have $ra < t < t+1 < (r+1)a$, so dividing by $a > 0$ yields $r < t/a < (t+1)/a < r+1$. Similarly, there exists an integer s with $s < t/b < (t+1)/b < s+1$, and by adding these two sets of inequalities we obtain $r+s < t/a + t/b < (t+1)/a + (t+1)/b < r+s+2$. Since $(1/a)+(1/b) = 1$, the last chain of inequalities simplifies to $r+s < t < t+1 < r+s+2$. But there is only one integer strictly contained between $r+s$ and $r+s+2$, so this is a contradiction. Therefore we must have $t = [ma]$ or $t = [mb]$ for some integer m.

5. Some of the people attending a party hate each other, but no one at the party hates more than three other guests. Prove that it is possible for all of the people at the party to assemble in two (large) rooms so that in each room, no individual hates more than one other person in that room. Assume that the hatred relation is symmetric, which means that if P hates Q, then also Q hates P.

SOLUTION. Start by randomly dividing the guests into the two rooms. Measure the intra-room hatred by counting the total number N of pairs of people who are in the same room and who hate each other. If there is someone who hates more people in his own room than in the other room, choose one such person and move him to the other room. Note that this move causes the quantity N is decrease.

We continue for as long as we can to move one person at a time from one room to the other, reducing N with each move. At each stage, N is an integer, and since it can never be negative, we cannot continue to reduce it forever. Eventually we must arrive at a situation where no move will reduce N. When that happens, no one will hate more people in his own room than in the other room. Since each person hates at most three people in total, we see that when N has been reduced as far as possible, no one will hate as many as two people in his own room.