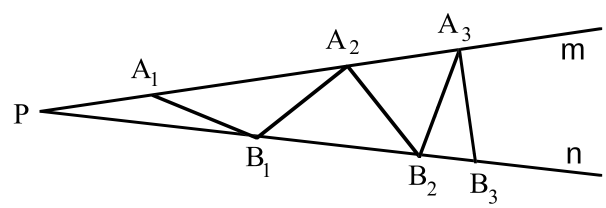
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2003-2004)

1. If x is any real number, we write $[x]$ to denote the largest integer that does not exceed x. Suppose that a and b are positive numbers such that $(1/a) + (1/b) = 1$. If m and n are positive integers such that $[ma] = [nb]$, show that ma and nb are integers.

SOLUTION. Let t be the integer with $t = [ma] = [nb]$. Then $t \le ma < t + 1$, so $t/a \le m < (t + 1)/a$. Similarly, $t/b \le n < (t + 1)/b$. By adding the last two inequalities and using $(1/a) + (1/b) = 1$, we obtain $t \le m + n < t + 1$. But $m + n$ is an integer, so we must have $t = m + n$ and hence all of the $\le$ inequalities above must be equalities. In particular, ma and nb are both equal to the integer t.

2. Lines m and n meet at point P as shown, and we perform the following construction. Point A_1 is chosen on line m so that the distance $PA_1 = 1$. Then a point B_1, different from P, is chosen on line n so that $A_1B_1 = 1$. Next, point A_2, different from A_1, is chosen on m so that $B_1A_2 = 1$, and similarly B_2 is chosen on n with $A_2B_2 = 1$. This process is continued for as long as possible, with each point farther from P than the previous one. Note that if $\angle P = 60°$, then $PA_1 = PB_1$. (a) If $PA_2 = PB_2$, compute $\angle P$. (b) If $PA_{10} = PB_{10}$, compute $\angle P$.



SOLUTION. Suppose that $\angle P$ equals α degrees. We proceed to compute some of the other angles in the figure. Since $\triangle PA_1B_1$ is isosceles, it follows that $\angle PB_1A_1 = \angle P = \alpha$. Next, $\angle B_1A_1A_2$ is an exterior angle for $\triangle PA_1B_1$, and so it is equal to the sum of the two "remote" interior angles of this triangle, and thus we have $\angle B_1A_1A_2 = 2\alpha$. Since $\triangle A_1B_1A_2$ is isosceles, it follows that also $\angle PA_2B_1 = 2\alpha$. Next, we compute $\angle A_2B_1B_2$ by observing that this angle is exterior to $\triangle PA_2B_1$, and so $\angle A_2B_1B_2 = \angle P + \angle PA_2B_1 = \alpha + 2\alpha = 3\alpha$. Using similar reasoning, we see that $\angle A_2B_2B_1 = 3\alpha$ since $\triangle B_1A_2B_2$ is isosceles, and thus $\angle A_3A_2B_2 = 3\alpha + \alpha = 4\alpha$ because this angle is exterior for $\triangle PA_2B_2$.

Continuing in this manner, we repeatedly obtain angles that are multiples of α. We have $\angle A_2A_1B_1 = 2\alpha$, $\angle A_3A_2B_2 = 4\alpha$, $\angle A_4A_3B_3 = 6\alpha$, and so on, with increasing even multiples of α as we move away from P on line m. It is easy to see that the general formula is $\angle A_{k+1}A_kB_k = 2k\alpha$. Now along line n, we have $\angle B_2B_1A_2 = 3\alpha$, $\angle B_3B_2A_3 = 5\alpha$, and the general formula here is $\angle B_{k+1}B_kA_{k+1} = (2k + 1)\alpha$.

Finally, we see that the condition for PA_k and PB_k to be equal is that $\angle PA_kB_k = \angle PB_kA_k$. We have $\angle PA_kB_k = 180 - \angle A_{k+1}A_kB_k = 180 - 2k\alpha$. Also, $\angle PB_kA_k = \angle B_kB_{k-1}A_k = (2(k-1)+1)\alpha$ because $\triangle B_{k-1}A_kB_k$ is isosceles. The condtion for equality of PA_k and PB_k is therefore $180 - 2k\alpha = (2k - 1)\alpha$, which simplifies to $\alpha = 180/(4k - 1)$. (As a check, note that if we set $k = 1$, we get $\alpha = 60$, which we know to be correct.) If $k = 2$, we get $\angle P = \alpha = 180/7$ and if $k = 10$, we get $\angle P = \alpha = 180/39$.

3. Suppose that $an^3 + bn^2 + cn + d$ is an integer for every integer $n \ge 1,000,000$. Prove that all of the numbers $6a$, $6b$, $6c$ and $6d$ are integers.

SOLUTION. If $f(n)$ is an integer for all large n, then the same is true for its first difference $f(n+1)-f(n)$. We start with the linear case where $f_1(n)=an+b$. If this is an integer for all large n, then its first difference $f_1(n+1)-f_1(n)=a(n+1)+b-(an+b)=a$ is also an integer. In particular, an is an integer, and hence so is b.

Next, in the quadratic case, suppose $f_2(n)=an^2+bn+c$ is an integer for all large n. Since the first difference of $f_2(n)$ is equal to $2an+(a+b)$, the linear result implies that $2a$ and $a+b$ are both integers.

Finally, in the cubic case, where $f_3(n)=an^3+bn^2+cn+d$ is an integer for all large n, the first difference is given by $3an^2+(3a+2b)n+(a+b+c)$. Thus, the quadratic result implies that $6a$ and $3a+(3a+2b)$ are both integers, and hence $2b$ is also an integer. Now look at $6f_3(n)$. By the above, the first two terms are integers and so $6cn+6d$ is also an integer for all large n. By an appeal to the linear case, we can now conclude that $6c$ and $6d$ are integers, and we have therefore shown that $6a$, $2b$, $6c$ and $6d$ are all integers.

4. Find all numbers a such that the equation $\Big|x+|x|+a\Big|+\Big|x-|x|-a\Big|=2$ has exactly three solutions.

SOLUTION. If x is a solution to the equation $\Big|x+|x|+a\Big|+\Big|x-|x|-a\Big|=2$, then

$$\Big|(-x)+|(-x)|+a\Big|+\Big|(-x)-|(-x)|-a\Big|=\Big|x-|x|-a\Big|+\Big|x+|x|+a\Big|=2,$$

and hence $-x$ is also a solution. Thus, in order for the equation to have an odd number of solutions, it is clear that $x=0$ must be one of these. Substituting $x=0$ into the equation, we obtain $2=|a|+|(-a)|=2|a|$, and hence $a=\pm1$. If $a=1$ and $x\ge0$, then the equation becomes $2=|2x+1|+1=2x+2$, so $x=0$. But $-x$ is a solution if and only if x is, so there is only one solution in this case. On the other hand, if $a=-1$ and $x\ge0$, then the equation becomes $|2x-1|+1=2$, so $|2x-1|=1$ and $x=0$ or 1. Thus, in this case, we have the three solutions $x=0,1,-1$, and therefore $a=-1$ is the unique number for which the equation has exactly three solutions.

5. Let x, y and z be positive. Prove that

$$(x^2+y^2+z^2)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge 3(x+y+z)\,.$$

SOLUTION. We begin by observing that $x^2-2xy+y^2=(x-y)^2\ge0$, which yields $x^2-xy+y^2\ge xy$. Since x and y are positive, we can multiply this inequality by $x+y$, and deduce that $x^3+y^3=(x+y)(x^2-xy+y^2)\ge(x+y)xy$. Again using the fact that x and y are positive, we divide by xy to obtain $(x^2/y)+(y^2/x)\ge x+y$. Of course, we also have similar inequalities relating x and z and also y and z. If we add these three inequalities we obtain

$$\frac{x^2}{y}+\frac{y^2}{x}+\frac{x^2}{z}+\frac{z^2}{x}+\frac{y^2}{z}+\frac{z^2}{y}\ge 2(x+y+z)\,.$$

Now look at the left side of the inequality we are trying to prove. If we multiply out the parentheses, we get nine terms, of which three are x, y and z. The remaining six terms form the left side of the inequality we have just established, and so their sum is at least $2(x+y+z)$. The entire left side of the desired inequality is therefore at least $3(x+y+z)$.