1. If \( x \) is any real number, we write \( \lfloor x \rfloor \) to denote the largest integer that does not exceed \( x \). Suppose that \( a \) and \( b \) are positive numbers such that \( (1/a) + (1/b) = 1 \). If \( m \) and \( n \) are positive integers such that \( \lfloor ma \rfloor = \lfloor nb \rfloor \), show that \( ma \) and \( nb \) are integers.

**SOLUTION.** Let \( t \) be the integer with \( t = \lfloor ma \rfloor = \lfloor nb \rfloor \). Then \( t \leq ma < t + 1 \), so \( t/a \leq m < (t + 1)/a \). Similarly, \( t/b \leq n < (t + 1)/b \). By adding the last two inequalities and using \( (1/a) + (1/b) = 1 \), we obtain \( t \leq m + n < t + 1 \). But \( m + n \) is an integer, so we must have \( t = m + n \) and hence all of the \( \leq \) inequalities above must be equalities. In particular, \( ma \) and \( nb \) are both equal to the integer \( t \).

2. Lines \( m \) and \( n \) meet at point \( P \) as shown, and we perform the following construction. Point \( A_1 \) is chosen on line \( m \) so that the distance \( PA_1 = 1 \). Then a point \( B_1 \), different from \( P \), is chosen on line \( n \) so that \( A_1B_1 = 1 \). Next, point \( A_2 \), different from \( A_1 \), is chosen on \( m \) so that \( B_1A_2 = 1 \), and similarly \( B_2 \) is chosen on \( n \) with \( A_2B_2 = 1 \). This process is continued for as long as possible, with each point farther from \( P \) than the previous one. Note that if \( \angle P = 60^\circ \), then \( PA_1 = PB_1 \). (a) If \( PA_2 = PB_2 \), compute \( \angle P \). (b) If \( PA_{10} = PB_{10} \), compute \( \angle P \).

**SOLUTION.** Suppose that \( \angle P \) equals \( \alpha \) degrees. We proceed to compute some of the other angles in the figure. Since \( \triangle PA_1B_1 \) is isosceles, it follows that \( \angle PB_1A_1 = \angle P = \alpha \). Next, \( \angle B_1A_1A_2 \) is an exterior angle for \( \triangle PA_1B_1 \), and so it is equal to the sum of the two “remote” interior angles of this triangle, and thus we have \( \angle B_1A_1A_2 = 2\alpha \). Since \( \triangle A_1B_1A_2 \) is isosceles, it follows that also \( \angle PA_2B_1 = 2\alpha \). Next, we compute \( \angle A_2B_1B_2 \) by observing that this angle is exterior to \( \triangle PA_2B_1 \), and so \( \angle A_2B_1B_2 = \angle P + \angle PA_2B_1 = \alpha + 2\alpha = 3\alpha \). Using similar reasoning, we see that \( \angle A_2B_2B_3 = 3\alpha \) since \( \triangle A_1B_2B_3 \) is isosceles, and thus \( \angle A_2A_3B_3 = 3\alpha + \alpha = 4\alpha \) because this angle is exterior for \( \triangle PA_2B_2 \).

Continuing in this manner, we repeatedly obtain angles that are multiples of \( \alpha \). We have \( \angle A_2A_1B_1 = 2\alpha \), \( \angle A_3A_2B_2 = 4\alpha \), \( \angle A_4A_3B_3 = 6\alpha \), and so on, with increasing even multiples of \( \alpha \) as we move away from \( P \) on line \( m \). It is easy to see that the general formula is \( \angle A_{k+1}A_kB_k = 2k\alpha \). Now along line \( n \), we have \( \angle B_2B_1A_2 = 3\alpha \), \( \angle B_3B_2A_3 = 5\alpha \), and the general formula here is \( \angle B_{k+1}B_kA_{k+1} = (2k + 1)\alpha \).

Finally, we see that the condition for \( PA_k \) and \( PB_k \) to be equal is that \( \angle PA_kB_k = \angle PB_kA_k \). We have \( \angle PA_kB_k = 180 - \angle A_{k+1}A_kB_k = 180 - 2k\alpha \). Also, \( \angle PB_kA_k = \angle B_kB_{k-1}A_k = (2(k-1) + 1)\alpha \) because \( \triangle B_{k-1}A_kB_k \) is isosceles. The condition for equality of \( PA_k \) and \( PB_k \) is therefore \( 180 - 2k\alpha = (2k-1)\alpha \), which simplifies to \( \alpha = 180/(4k-1) \).

(As a check, note that if we set \( k = 1 \), we get \( \alpha = 60 \), which we know to be correct.) If \( k = 2 \), we get \( \alpha = 180/7 \) and if \( k = 10 \), we get \( \alpha = 180/39 \).

3. Suppose that \( an^3 + bn^2 + cn + d \) is an integer for every integer \( n \geq 1,000,000 \). Prove that all of the numbers \( 6a, 6b, 6c \) and \( 6d \) are integers.
**SOLUTION.** If \( f(n) \) is an integer for all large \( n \), then the same is true for its first difference \( f(n+1) - f(n) \). We start with the linear case where \( f_1(n) = an + b \). If this is an integer for all large \( n \), then its first difference \( f_1(n+1) - f_1(n) = a(n+1) + b - (an+b) = a \) is also an integer. In particular, \( an \) is an integer, and hence so is \( b \).

Next, in the quadratic case, suppose \( f_2(n) = an^2 + bn + c \) is an integer for all large \( n \). Since the first difference of \( f_2(n) \) is equal to \( 2an + (a + b) \), the linear result implies that \( 2a \) and \( a + b \) are both integers.

Finally, in the cubic case, where \( f_3(n) = an^3 + bn^2 + cn + d \) is an integer for all large \( n \), the first difference is given by \( 3an^2 + (3a + 2b)n + (a + b + c) \). Thus, the quadratic result implies that \( 6a \) and \( 3a + (3a + 2b) \) are both integers, and hence \( 2b \) is also an integer. Now look at \( 6f_3(n) \). By the above, the first two terms are integers and so \( 6cn + 6d \) is also an integer for all large \( n \). By an appeal to the linear case, we can now conclude that \( 6c \) and \( 6d \) are integers, and we have therefore shown that \( 6a \), \( 2b \), \( 6c \) and \( 6d \) are all integers.

4. Find all numbers \( a \) such that the equation \(|x + |x| + a| + |x - |x| - a| = 2 \) has exactly three solutions.

**SOLUTION.** If \( x \) is a solution to the equation \(|x + |x| + a| + |x - |x| - a| = 2\), then \[
\begin{align*}
(|x|) + |(-x)| + a & + |(-x) - |(-x)| - a| = |x - |x| - a| + |x + |x| + a| = 2,
\end{align*}
\] and hence \( -x \) is also a solution. Thus, in order for the equation to have an odd number of solutions, it is clear that \( x = 0 \) must be one of these. Substituting \( x = 0 \) into the equation, we obtain \( 2 = |a| + |(-a)| = 2|a| \), and hence \( a = \pm 1 \). If \( a = 1 \) and \( x \geq 0 \), then the equation becomes \( 2 = |2x + 1| + 1 = 2x + 2 \), so \( x = 0 \). But \( -x \) is a solution if and only if \( x \) is, so there is only one solution in this case. On the other hand, if \( a = -1 \) and \( x \geq 0 \), then the equation becomes \( |2x - 1| + 1 = 2 \), so \( |2x - 1| = 1 \) and \( x = 0 \) or 1. Thus, in this case, we have the three solutions \( x = 0, 1, -1 \), and therefore \( a = -1 \) is the unique number for which the equation has exactly three solutions.

5. Let \( x, y \) and \( z \) be positive. Prove that
\[
(x^2 + y^2 + z^2) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \geq 3(x + y + z).
\]

**SOLUTION.** We begin by observing that \( x^2 - 2xy + y^2 = (x - y)^2 \geq 0 \), which yields \( x^2 - xy + y^2 \geq xy \). Since \( x \) and \( y \) are positive, we can multiply this inequality by \( x + y \), and deduce that \( x^3 + y^3 = (x + y)(x^2 - xy + y^2) \geq (x + y)xy \). Again using the fact that \( x \) and \( y \) are positive, we divide by \( xy \) to obtain \( (x^2/y) + (y^2/x) \geq x + y \). Of course, we also have similar inequalities relating \( x \) and \( z \) and also \( y \) and \( z \). If we add these three inequalities we obtain
\[
\frac{x^2}{y} + \frac{y^2}{x} + \frac{x^2}{z} + \frac{z^2}{x} + \frac{y^2}{z} + \frac{z^2}{y} \geq 2(x + y + z).
\]
Now look at the left side of the inequality we are trying to prove. If we multiply out the parentheses, we get nine terms, of which three are \( x, y \) and \( z \). The remaining six terms form the left side of the inequality we have just established, and so their sum is at least \( 2(x + y + z) \). The entire left side of the desired inequality is therefore at least \( 3(x + y + z) \).