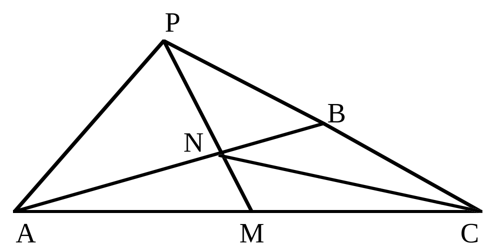
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2003-2004)

1. Find all positive integers n such that the number $n^4 + n^3 - 8$ is a perfect square. Prove that you have found all possibilities.

SOLUTION. If $n = 2$, then $n^4 + n^3 - 8 = 4^2$ is a perfect square and if $n = 3$, then $n^4 + n^3 - 8 = 10^2$ is a perfect square. We show that these are the only possibilities. Note that $n = 1$ does not yield a perfect square, so we can assume now that $n \geq 4$. Since $n^3 - 8 > 0$, we have $n^4 + n^3 - 8 > n^4 = (n^2)^2$. In particular, if $n^4 + n^3 - 8 = m^2$ is a perfect square (with $m \geq 0$), then $m > n^2$ and we can write $m = n^2 + k$ for some integer $k \geq 1$. Then $n^4 + n^3 - 8 = m^2 = (n^2 + k)^2 = n^4 + 2n^2k + k^2$ yields $n^2(n - 2k) = k^2 + 8$. This shows that $n - 2k > 0$ so $n > 2k$ and $n^2 > 4k^2$. Also, since $n - 2k \geq 1$, we have $(n^2/4) + 8 > k^2 + 8 = n^2(n - 2k) \geq n^2$, and thus $8 > 3n^2/4$. This is a contradiction since $n \geq 4$, by assumption, and hence $3n^2/4 \geq 12$.

2. In the figure, point M lies on side $\overline{AC}$ of $\triangle ABC$ and point N lies on side $\overline{AB}$. These points are chosen so that the three small triangles $\triangle AMN$, $\triangle CMN$ and $\triangle CBN$ all have equal area. If line segments $\overline{MN}$ and $\overline{BC}$ are extended to meet at point P, show that B is the midpoint of $\overline{PC}$.



SOLUTION. Draw $\overline{PA}$ as shown, and let a denote the area of each of the triangles $\triangle AMN$, $\triangle CMN$ and $\triangle CBN$. Since the area of a triangle is equal to "one-half base times height", we see that if two triangles have a common altitude, then their areas are proportional to the lengths of the corresponding bases. For example, we note that $\overline{CN}$ divides $\triangle ABC$ into two two smaller triangles having areas a and $2a$. These two triangles have a common altitude to vertex C, and thus the length of $\overline{AN}$ must be twice the length of $\overline{NB}$. Similarly, $\triangle PAN$ and $\triangle PNB$ have a common altitude to vertex P, and therefore it follows that the area of $\triangle PAN$ is twice the area of $\triangle PNB$. In particular, if we let b denote the area of $\triangle PNB$, then the area of $\triangle PAN$ is $2b$.

Now $\overline{NM}$ divides $\triangle ANC$ into two triangles with equal areas, and since these two small triangles have a common altitude to vertex N, it follows that their bases are equal and we have $AM = MC$. Therefore the areas of $\triangle PAM$ and $\triangle PCM$ are also equal. In other words, $2b + a = 2a + b$, and hence $a = b$. We now see that the areas of $\triangle PNB$ and $\triangle CNB$ are equal, and since these triangles have a common altitude to vertex N, they have equal bases. We conclude that $PB = BC$, as wanted.

3. In the previous problem set, we were introduced to the strange spelling rules on the planet AZAAZ. In their language, the letter A can be replaced by ZAZ and the combination ZAZ can be replaced by A, and these transformations can be made as many times as desired. We saw that AAAZZ is a valid spelling for the name of the planet, but that ZAZAZ is not. Decide if AZAZA is a valid spelling for the name of the planet and prove that your answer is correct.

SOLUTION. In the planet AZAAZ, the words are read from left to right. Thus the first letter of the planet name is an A, the second is a Z, and so on. We wish to keep track of the number $n = n(w)$ of Zs in a word w that occur in an even position. The above planet name has one Z in position 2 and one Z in position 5, so for this word, we have $n = 1$.

Let us see what happens to $n(w)$ if we replace some A by ZAZ to obtain a new spelling w'. First, the positions of the original Zs to the left of this A are unchanged, so they yield the same contribution to $n(w')$ as they did to $n(w)$. The positions of the original Zs to the right of A are increased by two, since we added two new letters to the word. Thus we still get the same number of Zs in even positions and hence the same contribution to $n(w')$. Finally, the two new Zs are two apart, so they are either both in an even position or both in an odd position. We conclude that $n(w') = n(w)$ or $n(w) + 2$. Similarly, if we replace some ZAZ in w by A to obtain a new spelling w'', then $n(w'') = n(w)$ or $n(w) - 2$.

In other words, if $n(w)$ is odd, then it remains odd under all sequences of replacements. As we saw, the planet name AZAAZ has $n = 1$. So all correct spellings will have an odd n. But AZAZA has its two Zs in even positions, so it cannot be a correct spelling.

4. If x and y are any two integers, let $x \,\square\, y$ denote an integer determined by x and y. Assume that the binary operation $\square$ satisfies the rules: $x \,\square\, (y+z) = (x \,\square\, y) - z$, $(y+z) \,\square\, x = (y \,\square\, x) + 2z$, and $1 \,\square\, 1 = 1$. Compute $25 \,\square\, 10$.

SOLUTION. Let b be any integer and put $x = 1$, $y = 1$ and $z = b - 1$ in the rule $x \,\square\, (y+z) = (x \,\square\, y) - z$. Since $1 \,\square\, 1 = 1$, we obtain $1 \,\square\, b = 1 \,\square\, \big(1 + (b-1)\big) = (1 \,\square\, 1) - (b-1) = 1 - (b-1) = 2 - b$. Next, let a and b be any integers and put $x = b$, $y = 1$ and $z = a - 1$ in the rule $(y+z) \,\square\, x = (y \,\square\, x) + 2z$. Using $1 \,\square\, b = 2 - b$, we obtain $a \,\square\, b = \big(1 + (a-1)\big) \,\square\, b = (1 \,\square\, b) + 2(a-1) = (2-b) + 2(a-1) = 2a - b$. In particular, $25 \,\square\, 10 = 2{\cdot}25 - 10 = 40$.

5. Find all positive real numbers x, y and z that satisfy the following three equations: $x + \dfrac{4}{xy} = 3$, $y + \dfrac{4}{yz} = 3$, and $z + \dfrac{4}{zx} = 3$.

SOLUTION. It is no loss to assume that x is the largest of the three unknowns. In particular, $x \geq y$ and hence $zx \geq zy$, since $z > 0$. Thus $4/(zx) \leq 4/(zy)$ and this yields $z = 3 - \dfrac{4}{zx} \geq 3 - \dfrac{4}{yz} = y$. We now have $zx \geq yx$, so $4/(zx) \leq 4/(yx)$ and consequently $z = 3 - \dfrac{4}{zx} \geq 3 - \dfrac{4}{xy} = x$. But $x \geq z$ by assumption, and therefore $x = z$.

If we substitute $x = z$ into the third of the given equations and simplify, we obtain $z^3 - 3z^2 + 4 = 0$. A bit of experimentation shows that $z = 2$ is one solution to this cubic equation, and thus $z - 2$ must be a factor of the polynomial $z^3 - 3z^2 + 4$. Using division to find the other factor, we see that $z^3 - 3z^2 + 4 = (z-2)(z^2 - z - 2)$, and so we need to solve the quadratic equation $z^2 - z - 2 = 0$ in order to find all other possibilities for z. Since $z^2 - z - 2 = (z-2)(z+1)$, we see that $z = 2$ is the only positive solution to the original cubic equation, and thus $x = z = 2$. Finally, we deduce from the first of the given equations that also $y = 2$.