WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

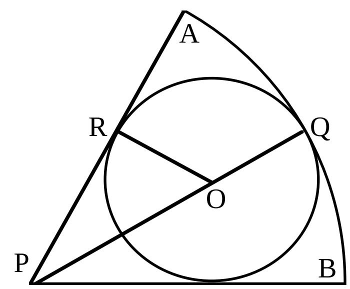
SOLUTIONS TO PROBLEM SET I (2003-2004)

1. Suppose that n is a positive integer with the property that there are exactly six different positive integers m such that n/m is an integer. If one of these six numbers is $m = 27$, find all possibilities for n.

SOLUTION. We can factor n into powers of prime numbers, and obtain $n = p^a q^b \cdots$, where $p, q, \ldots$ are different prime numbers and where the exponents are positive integers. If m is a positive integer such that n/m is an integer, then all prime divisors of m must divide n and the exponent with which each prime occurs in the factorization of m is at most the exponent to which that prime occurs in n. Such a number m can therefore be written in the form $p^x q^y \cdots$, where $0 \le x \le a$, $0 \le y \le b$ and so on. Thus there are $a+1$ possibilities for x, $b+1$ possibilities for y and continuing in this manner, we see that there are $(a+1)(b+1)\cdots$ possible numbers m. Note that each of the factors here is at least 2.

We are given that there are exactly 6 possible numbers m, and clearly there are only two ways to obtain 6 as a product of numbers that are at least 2. Either there is just one factor equal to 6 or there are exactly two factors, namely 2 and 3. These factors are 1 more than the exponents in the prime factorization of n, and thus the only possibilities for n are $n = p^5$, the fifth power of some prime p, or $n = pq^2$ where p and q are distinct primes. Now we are told that n is a multiple of $27 = 3^3$, and this eliminates the possibility that $n = pq^2$. We conclude therefore that $n = 3^5 = 243$.

2. In the figure, APB is a sector of a circle centered at P. This means that $\overline{PA}$ and $\overline{PB}$ are radii and that AB is an arc of the circle. A smaller circle is inscribed in the sector, as shown, so that it is tangent to $\overline{PA}$, $\overline{PB}$ and arc AB. Given that $\angle APB = 60°$, find the fraction of the area of the sector that is covered by the inscribed circle.



SOLUTION. Draw line segment $\overline{PQ}$, where Q is the point of tangency of the circle and the arc, as shown, and note that $\overline{PQ}$ must go through the center O of the circle and that it bisects $\angle APB$. Next, draw radius $\overline{OR}$, as shown, where R is the point of tangency of the circle with $\overline{PA}$. Note that $\angle ORP = 90°$ and $\angle OPR = 30°$.

Let $r = OR$, so that the area of the circle is πr^2. Since $\triangle ORP$ is a 30°–60°–90° triangle, its hypotenuse $\overline{PO}$ is twice as long as its shorter side $\overline{OR}$, and thus $PO = 2r$. Since $\overline{OQ}$ is a radius of the circle, we have $OQ = r$ and $PQ = 3r$. Thus the area of the big circle centered at P is $9\pi r^2$. The area of the 60°-sector is one sixth of this, which is $3\pi r^2/2$, and hence the fraction of the sector covered by the circle is $(\pi r^2)/(3\pi r^2/2) = 2/3$.

3. Suppose that $a^2 + b^2 + c^2$ is a multiple of 16, where a, b and c are integers. Show that $a^3 + b^3 + c^3$ is a multiple of 64.

SOLUTION. First, let us see what we can say if $a^2 + b^2 + c^2$ is a multiple of 4. Since the sum is even, there are two possibilities. Either a^2, b^2 and c^2 are all even or exactly two of them are odd. But an even square is a multiple of 4 and an odd square is 1 more than a multiple of 4. (Actually, it is 1 more than a multiple of 8, but we shall not need that additional information.) If not all of a^2, b^2 and c^2 are even, it follows that one of these numbers is a multiple of 4 and two of them are each 1 more than a multiple of 4, and thus their sum is 2 more than a multiple of 4. But this is not the case since we are assuming that $a^2+b^2+c^2$ is a multiple of 4. We have therefore shown that if $a^2+b^2+c^2$ is a multiple of 4, then a, b and c are all even.

Now suppose that $a^2+b^2+c^2$ is a multiple of 16. In particular, this sum is a multiple of 4, and thus, as we have seen, a, b and c must be even and we can write $a = 2r$, $b = 2s$ and $c = 2t$. Now $4(r^2+s^2+t^2) = a^2+b^2+c^2$ is a multiple of 16, and hence $r^2+s^2+t^2$ is a multiple of 4. By the same reasoning as above, r, s and t are all even, and we can write $r = 2x$, $s = 2y$ and $t = 2z$. Then $a = 4x$, $b = 4y$ and $c = 4z$, and we see that $a^3+b^3+c^3 = 64(x^3+y^3+z^3)$ is indeed a multiple of 64.

4. Every word in the language of the planet AZAAZ can be spelled using just the letters A and Z, and most words can be spelled in many different ways. This is because the letter A can always be replaced by ZAZ and the combination ZAZ can always be replaced by A. These replacements can be done repeatedly, but no others are allowed. For example, the capital city of AZAAZ is ZAZA, and this name can also be spelled AA or AZAZ or ZAZZAZ, and there are many other possible spellings. Prove that AAAZZ is a correct spelling for the name of the planet, but that ZAZAZ is not.

SOLUTION. To find some alternative spellings for the word AZAAZ, we start by replacing the third A by ZAZ to obtain AZAZAZZ. Now the ZAZ beginning at the second position can be replaced by A, and this yields AAAZZ as another correct spelling for the name of the planet.

To prove that ZAZAZ is not a correct spelling for this name, we observe that changing A to ZAZ increases the total number of Zs by 2 and replacing ZAZ by A decreases the total number of Zs by 2. Now there are precisely two *Zs* in the original spelling AZAAZ and since we are always adding or subtracting 2, the number of Zs must remain even no matter what we do. Since ZAZAZ contains an odd number of Zs, it is impossible to obtain that spelling by any sequence of legal substitutions.

5. Decide whether or not the number $a = \sqrt[3]{99-70\sqrt{2}} + \sqrt{8}$ is an integer and prove that your answer is correct.

SOLUTION. Using a calculator, we see that a is *approximately* 3, and we work to prove that a is, in fact, exactly 3. We compute that

$$\begin{aligned}(3-\sqrt{8})^3 &= 3^3 - 3(3^2)\sqrt{8} + 3(3)(\sqrt{8})^2 - (\sqrt{8})^3 \\ &= 27 - 27\sqrt{8} + 72 - 8\sqrt{8} = 99 - 35\sqrt{8} = 99 - 70\sqrt{2}\,.\end{aligned}$$

Thus $\sqrt[3]{99-70\sqrt{2}} = 3-\sqrt{8}$, and so $a = \sqrt[3]{99-70\sqrt{2}} + \sqrt{8} = 3$ is an integer.