WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2002-2003)

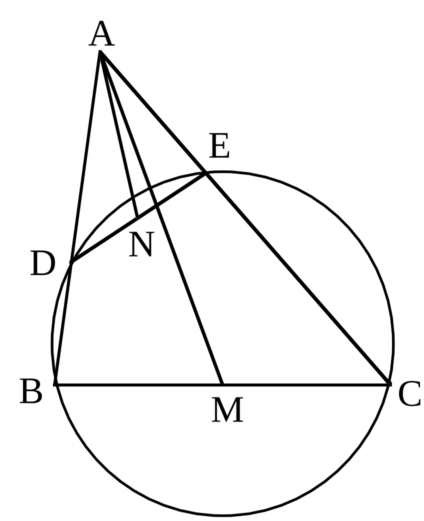
1. An integer n is a sum of two squares if there are integers x, y (possibly 0) with $n = x^2 + y^2$. If each of the integers a and b is a sum of two squares, show that their product ab is also a sum of two squares. Prove that the number $5^{64} - 3^{64}$ is a sum of two squares.

SOLUTION. Suppose that the integers a and b are each a sum of two squares, so that $a = x^2 + y^2$ and $b = u^2 + v^2$ for suitable integers x, y, u, v. Then we have

$$ab = (x^2 + y^2)(u^2 + v^2) = (xu - yv)^2 + (xv + yu)^2,$$

so ab is also a sum of two squares. Now we consider $5^{64} - 3^{64}$, and note that $64 = 2^6$ is a power of 2. We will show that $5^k - 3^k$ is a sum of two squares for any exponent k which is a power of 2. To start with, we have $5^1 - 3^1 = 1^2 + 1^2$ and $5^2 - 3^2 = 4^2 + 0^2$. Next, $5^4 - 3^4 = (5^2 - 3^2)(5^2 + 3^2)$ and, since each factor is a sum of two squares, it follows that $5^4 - 3^4$ is also a sum of two squares. As we will see, this same trick applies more generally. Indeed, suppose we already know that $5^{2^n} - 3^{2^n}$ is a sum of two squares for some integer $n \geq 1$. Then we have $5^{2^{n+1}} - 3^{2^{n+1}} = (5^{2^n} - 3^{2^n})(5^{2^n} + 3^{2^n})$, and observe that $(5^{2^n} + 3^{2^n})$ is a sum of two squares since 2^n is even. Thus each of the preceding two factors is a sum of two squares and hence the same is true of $5^{2^{n+1}} - 3^{2^{n+1}}$. It follows by mathematical induction that $5^k - 3^k$ is a sum of two squares whenever $k = 2^n$ is a power of 2.

2. A circle is drawn through vertices B and C of $\triangle ABC$, and this circle meets sides $\overline{AB}$ and $\overline{AC}$ at points D and E, as shown. If the midpoint of $\overline{BC}$ is M and the midpoint of $\overline{DE}$ is N, show that $\angle DAN = \angle CAM$.



SOLUTION. Since quadrilateral $DECB$ is inscribed in a circle, its opposite angles are supplementary, and it follows that $\angle C = 180^\circ - \angle EDB = \angle ADE$. In particular, two angles of $\triangle ADE$ are equal to two angles of $\triangle ACB$, so these two triangles are similar and hence $AD/AC = DE/CB$. Of course, $DN = \frac{1}{2}DE$ and $CM = \frac{1}{2}CB$, so $DE/CB = DN/CM$, and thus $AD/AC = DN/CM$. In addition, we know that $\angle ADN = \angle ACM$, so it follows by the SAS similarity criterion that $\triangle ADN$ is similar to $\triangle ACM$. We can now conclude that $\angle DAN = \angle CAM$, since these are corresponding angles of similar triangles.

3. The cubic polynomial $p(x) = x^3 - 4x^2 + 2$ has three distinct real roots, say a, b and c. Find $a^4 + b^4 + c^4$.

SOLUTION. For every integer $k \geq 0$, let $s_k = a^k + b^k + c^k$ denote the sum of the kth powers of a, b and c, and note that $s_0 = a^0 + b^0 + c^0 = 3$. Next, since a, b and c are the distinct roots of $p(x)$, it is clear that

$$x^3 - 4x^2 + 2 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc.$$

In particular, $s_1 = a + b + c = 4$ and $ab + bc + ca = 0$. Thus $16 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = s_2 + 0$, and hence $s_2 = 16$. The remaining s_k's, with $k \geq 3$, can now be found simply in terms of the power sums with smaller exponents. To this end, note that $0 = p(a) = a^3 - 4a^2 + 2$, and by multiplying by a^{k-3}, we have $a^k = 4a^{k-1} - 2a^{k-3}$. Similarly, $b^k = 4b^{k-1} - 2b^{k-3}$ and $c^k = 4c^{k-1} - 2c^{k-3}$. By adding these three equations, we therefore obtain $s_k = 4s_{k-1} - 2s_{k-3}$. When $k = 3$, this yields $s_3 = 4s_2 - 2s_0 = 4{\cdot}16 - 2{\cdot}3 = 58$, and finally when $k = 4$, we see that $s_4 = 4s_3 - 2s_1 = 4{\cdot}58 - 2{\cdot}4 = 224$. Obviously, we can continue this process to compute power sums with larger and larger exponents.

4. Suppose that each positive integer is colored either red or blue. Show that there exist three positive integers $x < y < z$ having the same color and such that $z - y = y - x$.

SOLUTION. We will show, in fact, that the triple of positive integers $x < y < z$ can be chosen from the set $\{1, 2, \ldots, 13\}$. To start with, look at the numbers 5, 7 and 9. At least two of these must have the same color, which we can assume, without loss of generality, to be red. There are three cases to consider, according to which pair of numbers in the set $\{5, 7, 9\}$ is known to be red. Suppose first that 5 and 7 are red. If either 3, 6 or 9 is red, then one of the triples $3 < 5 < 7$, $5 < 6 < 7$ or $5 < 7 < 9$ is entirely red. Thus we can assume that 3, 6 and 9 are blue, and then $3 < 6 < 9$ is a blue triple. Next, suppose that 7 and 9 are red. Then the same argument applies, but shifted up by two numbers. Namely, if either 5, 8 or 11 is red, then at least one of the triples $5 < 7 < 9$, $7 < 8 < 9$ or $7 < 9 < 11$ is entirely red. Otherwise, $5 < 8 < 11$ is a blue triple. Finally, if 5 and 9 are red members of $\{5, 7, 9\}$, then we again apply the same argument, this time using skips twice as large. Specifically, if either 1, 7 or 13 is red, then at least one of the triples $1 < 5 < 9$, $5 < 7 < 9$ or $5 < 9 < 13$ is entirely red. Otherwise, $1 < 7 < 13$ is a blue triple. Note that the colors of the numbers 2, 4, 10 and 12 are irrelevant to this proof.

5. Alice and Bob play a game by taking turns removing some stones from a pile. The rules require that the number of stones removed at each turn must be either 1, 2, 3 or 4, and the winner of the game is the person who takes the last stone. If we start with 100 stones and Alice goes first, prove that Bob can win, no matter what Alice does.

SOLUTION. Bob's plan is to always to remove 5 minus the number of stones that Alice just removed. Thus, if Alice took 1 stone, then Bob takes 4, if Alice took 2 stones, then Bob takes 3, and so on. Suppose at some point, when it is Alice's turn, that the number of stones in the pile is positive and divisible by 5. (Note that this is precisely the situation at the beginning since the game starts with 100 stones.) Then there are at least 5 stones present and, since Alice can remove at most 4 of these, Alice cannot take the last stone. Now Bob moves, and by using his planned response to Alice's move, he will make the number of stones again be divisible by 5. Thus, either he takes the last stone or he leaves Alice with a positive number of stones divisible by 5. Continuing in this manner, we see that Alice can never take the last stone, and thus Bob must eventually win.