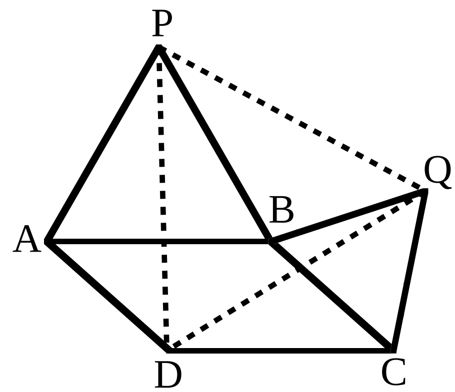
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2002-2003)

1. Find the smallest positive integer m such that m is not a square and yet in the decimal expansion of $\sqrt{m}$, the decimal point is followed by at least four consecutive zeros.

SOLUTION. If k is the integer part of $\sqrt{m}$, then by assumption we have $k < \sqrt{m} < k + .0001 = k + 1/10000$. Thus $k^2 < m < k^2 + (k/5000) + (1/10000)^2$. Since m is an integer, it follows that $(k/5000) + (1/10000)^2 \geq 1$, so $k \geq 5000$. The smallest m will then occur when $k = 5000$ and $m = k^2 + 1 = (5000)^2 + 1$.

2. Equilaterial triangles $\triangle PAB$ and $\triangle QBC$ are constructed on two sides of the parallelogram $ABCD$, as shown. Prove that $\triangle DPQ$ is also equilaterial.



SOLUTION. Since $ABCD$ is a parallelogram, its opposite sides are equal and so are its opposite angles. In particular, since the four angles of a quadrilateral sum to $360°$, we have $\angle DAB + \angle ABC = 180°$. We show now that $\triangle DAP$ and $\triangle QBP$ are congruent by SAS. To start with, since $\angle CBQ = \angle ABP = 60°$, we have $\angle QBP + \angle ABC + 2{\cdot}60° = 360°$ and hence $\angle QBP = 240° - \angle ABC = 60° + (180° - \angle ABC) = 60° + \angle DAB$. But $\angle PAB$ is also a $60°$ angle, so $\angle DAP = 60° + \angle DAB = \angle QBP$. Next, by using the fact that all sides of $\triangle APB$ and of $\triangle BQC$ are equal, we have $AP = BP$ and $AD = BC = BQ$. Thus $\triangle DAP \cong \triangle QBP$, as claimed. Hence $PD = PQ$, and we see that $\triangle DPQ$ is at least isosceles. Finally, since $\angle APD = \angle BPQ$, we have $\angle DPQ = \angle DPB + \angle BPQ = \angle DPB + \angle APD = \angle APB = 60°$, and $\triangle DPQ$ is indeed equilateral.

3. (New Year's Problem). Find the number of nonnegative integers n such that $2003 + n$ is a multiple of $n + 1$.

SOLUTION. If $2003 + n = m(n + 1)$, then $2002 = m(n + 1) - (n + 1) = (m - 1)(n + 1)$ and $n + 1 \geq 1$ is a divisor of $2002 = 2{\cdot}7{\cdot}11{\cdot}13$. In particular, $n + 1 = 2^a{\cdot}7^b{\cdot}11^c{\cdot}13^d$ with each of a, b, c, d being 0 or 1. Since there are $2^4 = 16$ possible choices for these exponents, there are 16 possible choices for n.

4. Let F_n be the n th Fibonacci number, so that $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ and in general, $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. Let S be a set consisting of finitely many different Fibonacci numbers F_s with $s \geq 2$. Assume that S has more than one member and that the sum of all of the members of S is a Fibonacci number. Show that S must contain both F_{k-1} and F_k for some integer k.

SOLUTION. We first observe that $F_1 + F_2 + \cdots + F_n < F_{n+2}$ for all $n \geq 1$. This is clear for $n = 1$. Now suppose the result holds for some integer $n \geq 1$. Then adding F_{n+1} to both sides of the previous inequality yields $F_1 + F_2 + \cdots + F_n + F_{n+1} < F_{n+1} + F_{n+2} = F_{n+3}$. Since

$n+3=(n+1)+2$, we see that the required inequality now holds for the integer $n+1$. It follows, by mathematical induction, that the inequality holds for all integers $n \geq 1$. Note that the Fibonacci numbers with subscripts ≥ 2 all have numerically different values.

We now show that the set S must contain both F_{k-1} and F_k for some integer k. We do this by mathematical induction on the number of members of S which we know is at least two. Let F_s be the largest member of S and let ΣS denote the sum of the members of S. Then $\Sigma S > F_s$, since F_s is not the only member of S, and $\Sigma S \leq F_2 + F_3 + \cdots + F_s < F_{s+2}$, where the second inequality holds by the observation of the previous paragraph. Thus, since ΣS is a Fibonacci number, by assumption, the only possibility is that $\Sigma S = F_{s+1}$. Now let T be the set of all members of S other than F_s. Then $F_{s+1} = \Sigma S = F_s + \Sigma T$, so $\Sigma T = F_{s+1} - F_s = F_{s-1}$. If T has only one member, then $T = \{F_{s-1}\}$ and $S = \{F_{s-1}, F_s\}$, as required. On the other hand, if T has at least two members, then we use the fact that T is a smaller set than S with the same property that ΣT is a Fibonacci number. By induction, T must contain F_{k-1} and F_k for some integer k. But every member of T is a member of S, so F_{k-1} and F_k are also members of S.

5. Let f be a function which assigns to each positive integer n a positive integer $f(n)$. We suppose that $f(ab) = f(a)f(b) - f(a) - f(b) + 2$ for all positive integers a, b and that $f(c!) = c! + 1$ for all $c \geq 10^{10}$. Show that $f(n) = n+1$ for all n.

SOLUTION. Let us say that a positive integer n is "good" if $f(n) = n+1$. Our goal, therefore, is to prove that all positive integers are good. First, we show that if n and m are positive integers and both n and nm are good, then m is also good. We have

$$nm + 1 = f(nm) = f(n)f(m) - f(n) - f(m) + 2 = (n+1)f(m) - (n+1) - f(m) + 2\,.$$

Solving for $f(m)$, we get $f(m) = m+1$, and thus m is good, as claimed.

If $c > 10^{10}$, then $c - 1 \geq 10^{10}$ and we know by assumption that both $(c-1)!$ and $c!$ are good. But $c! = (c-1)!{\cdot}c$, and so if we set $n = (c-1)!$ and $m = c$, our previous calculation guarantees that c is good. Now consider the set B of all positive integers that are not good, and assume that B is not the empty set. We have seen that no member of B can exceed 10^{10}, and so B contains some largest member b. Then $b+1$ is good, and so is $(b+1)b$. It therefore follows by our earlier observation that b is also good. But this is not true since we chose b in the set B. This contradiction proves that our assumption that B is nonempty must be wrong, and so B is empty. In other words, all positive integers are good.