1. Find the smallest positive integer \( m \) such that \( m \) is not a square and yet in the decimal expansion of \( \sqrt{m} \), the decimal point is followed by at least four consecutive zeros.

**SOLUTION.** If \( k \) is the integer part of \( \sqrt{m} \), then by assumption we have \( k < \sqrt{m} < k + .0001 = k + 1/10000 \). Thus \( k^2 < m < k^2 + (k/5000) + (1/10000)^2 \). Since \( m \) is an integer, it follows that \( (k/5000) + (1/10000)^2 \geq 1 \), so \( k \geq 5000 \). The smallest \( m \) will then occur when \( k = 5000 \) and \( m = k^2 + 1 = (5000)^2 + 1 \).

2. Equilateral triangles \( \triangle PAB \) and \( \triangle QBC \) are constructed on two sides of the parallelogram \( ABCD \), as shown. Prove that \( \triangle DPQ \) is also equilateral.

**SOLUTION.** Since \( ABCD \) is a parallelogram, its opposite sides are equal and so are its opposite angles. In particular, since the four angles of a quadrilateral sum to \( 360^\circ \), we have \( \angle DAB + \angle ABC = 180^\circ \). We show now that \( \triangle DAP \) and \( \triangle QBP \) are congruent by SAS. To start with, since \( \angle CBQ = \angle ABP = 60^\circ \), we have \( \angle QBP + \angle ABC + 2 \cdot 60^\circ = 360^\circ \) and hence \( \angle QBP = 240^\circ - \angle ABC = 60^\circ + (180^\circ - \angle ABC) = 60^\circ + \angle DAB \).

But \( \angle PAB \) is also a \( 60^\circ \) angle, so \( \angle DAP = 60^\circ + \angle DAB = \angle QBP \). Next, by using the fact that all sides of \( \triangle APB \) and of \( \triangle BQC \) are equal, we have \( AP = BP \) and \( AD = BC = BQ \). Thus \( \triangle DAP \cong \triangle QBP \), as claimed. Hence \( PD = PQ \), and we see that \( \triangle DPQ \) is at least isosceles. Finally, since \( \angle APD = \angle BQP \), we have \( \angle DPQ = \angle DAP + \angle BQP = \angle DPB + \angle APD = \angle APB = 60^\circ \), and \( \triangle DPQ \) is indeed equilateral.

3. (New Year’s Problem). Find the number of nonnegative integers \( n \) such that \( 2003 + n \) is a multiple of \( n + 1 \).

**SOLUTION.** If \( 2003 + n = m(n + 1) \), then \( 2002 = m(n + 1) - (n + 1) = (m - 1)(n + 1) \) and \( n + 1 \geq 1 \) is a divisor of \( 2002 = 2 \cdot 7 \cdot 11 \cdot 13 \). In particular, \( n + 1 = 2^a \cdot 7^b \cdot 11^c \cdot 13^d \) with each of \( a, b, c, d \) being 0 or 1. Since there are \( 2^4 = 16 \) possible choices for these exponents, there are 16 possible choices for \( n \).

4. Let \( F_n \) be the \( n \)th Fibonacci number, so that \( F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5 \) and in general, \( F_n = F_{n-1} + F_{n-2} \) for all \( n \geq 3 \). Let \( S \) be a set consisting of finitely many different Fibonacci numbers \( F_i \) with \( s \geq 2 \). Assume that \( S \) has more than one member and that the sum of all of the members of \( S \) is a Fibonacci number. Show that \( S \) must contain both \( F_{k-1} \) and \( F_k \) for some integer \( k \).

**SOLUTION.** We first observe that \( F_1 + F_2 + \cdots + F_n < F_{n+2} \) for all \( n \geq 1 \). This is clear for \( n = 1 \). Now suppose the result holds for some integer \( n \geq 1 \). Then adding \( F_{n+1} \) to both sides of the previous inequality yields \( F_1 + F_2 + \cdots + F_n + F_{n+1} < F_{n+1} + F_{n+2} = F_{n+3} \). Since
\[ n + 3 = (n + 1) + 2, \] we see that the required inequality now holds for the integer \( n + 1 \). It follows, by mathematical induction, that the inequality holds for all integers \( n \geq 1 \). Note that the Fibonacci numbers with subscripts \( \geq 2 \) all have numerically different values.

We now show that the set \( S \) must contain both \( F_{k-1} \) and \( F_k \) for some integer \( k \). We do this by mathematical induction on the number of members of \( S \) which we know is at least two. Let \( F_s \) be the largest member of \( S \) and let \( \Sigma S \) denote the sum of the members of \( S \). Then \( \Sigma S > F_s \), since \( F_s \) is not the only member of \( S \), and \( \Sigma S \leq F_2 + F_3 + \cdots + F_s < F_{s+2} \), where the second inequality holds by the observation of the previous paragraph. Thus, since \( \Sigma S \) is a Fibonacci number, by assumption, the only possibility is that \( \Sigma S = F_{s+1} \). Now let \( T \) be the set of all members of \( S \) other than \( F_s \). Then \( F_{s+1} = \Sigma S = F_s + \Sigma T \), so \( \Sigma T = F_{s+1} - F_s = F_{s-1} \). If \( T \) has only one member, then \( T = \{ F_{s-1} \} \) and \( S = \{ F_{s-1}, F_s \} \), as required. On the other hand, if \( T \) has at least two members, then we use the fact that \( T \) is a smaller set than \( S \) with the same property that \( \Sigma T \) is a Fibonacci number. By induction, \( T \) must contain \( F_{k-1} \) and \( F_k \) for some integer \( k \). But every member of \( T \) is a member of \( S \), so \( F_{k-1} \) and \( F_k \) are also members of \( S \).

5. Let \( f \) be a function which assigns to each positive integer \( n \) a positive integer \( f(n) \). We suppose that \( f(ab) = f(a)f(b) - f(a) - f(b) + 2 \) for all positive integers \( a, b \) and that \( f(c!) = c! + 1 \) for all \( c \geq 10^{10} \). Show that \( f(n) = n + 1 \) for all \( n \).

**SOLUTION.** Let us say that a positive integer \( n \) is “good” if \( f(n) = n + 1 \). Our goal, therefore, is to prove that all positive integers are good. First, we show that if \( n \) and \( m \) are positive integers and both \( n \) and \( nm \) are good, then \( m \) is also good. We have

\[
nm + 1 = f(nm) = f(n)f(m) - f(n) - f(m) + 2 = (n + 1)f(m) - (n + 1) - f(m) + 2.
\]

Solving for \( f(m) \), we get \( f(m) = m + 1 \), and thus \( m \) is good, as claimed.

If \( c > 10^{10} \), then \( c - 1 \geq 10^{10} \) and we know by assumption that both \((c - 1)!\) and \( c! \) are good. But \( c! = (c - 1)! \cdot c \), and so if we set \( n = (c - 1)! \) and \( m = c \), our previous calculation guarantees that \( c \) is good. Now consider the set \( B \) of all positive integers that are not good, and assume that \( B \) is not the empty set. We have seen that no member of \( B \) can exceed \( 10^{10} \), and so \( B \) contains some largest member \( b \). Then \( b + 1 \) is good, and so is \((b + 1)b \). It therefore follows by our earlier observation that \( b \) is also good. But this is not true since we chose \( b \) in the set \( B \). This contradiction proves that our assumption that \( B \) is nonempty must be wrong, and so \( B \) is empty. In other words, all positive integers are good.