WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2002-2003)

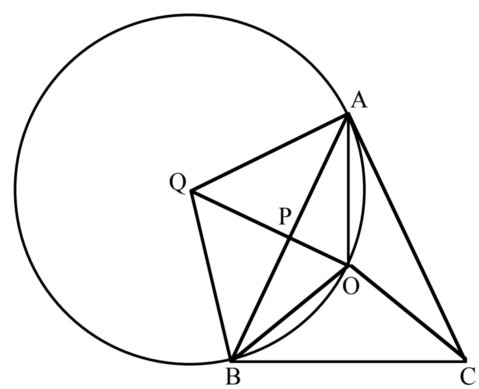
1. Find the smallest possible value for the expression $(x+y+z)(xy+xz+yz)/(xyz)$ as x, y and z run over all positive real numbers.

SOLUTION. Call the above expression E, and note that if $x=y=z$, then $E=(3x)(3x^2)/x^3=9$. We claim that 9 is in fact the smallest possible value for E, and for this, we must show that $E \geq 9$, in general. Equivalently, we must show that $(x+y+z)(xy+xz+yz) \geq 9xyz$ for all positive real numbers x, y, z. There are several proofs of this fact and we mention just two. The first is a consequence of the following easily verified formula.

$$(x+y+z)(xy+xz+yz) - 9xyz = x(y-z)^2 + y(z-x)^2 + z(x-y)^2 \geq 0$$

The second uses the *arithmetic-geometric mean inequality* which asserts that if $r_1, r_2, \ldots r_n$ are n positive real numbers, then their arithmetic mean is greater than or equal to their geometric mean, that is $(r_1+r_2+\cdots+r_n)/n \geq (r_1 r_2 \cdots r_n)^{1/n}$. Applying this to the numbers x, y, z and then to the numbers xy, yz, zx, we obtain $(x+y+z) \geq 3(xyz)^{1/3}$ and $(xy+yz+zx) \geq 3(xyz)^{2/3}$. Multiplying these two inequalities then yields $(x+y+z)(xy+xz+yz) \geq 9(xyz)$, as required.

2. Recall that it is always possible to draw a circle through the three vertices of a triangle. This is the *circumcircle* of the triangle and its center is called the *circumcenter* of the triangle. Suppose that point O is the circumcenter of the isosceles triangle $\triangle ABC$, where $AB = AC$. Show that line $\overline{AC}$ is tangent to the circumcircle of $\triangle AOB$.



SOLUTION. Let Q be the circumcenter of $\triangle AOB$ and draw the three radii $\overline{QA}$, $\overline{QB}$ and $\overline{QO}$. Of course, $QA = QB = QO$. Also draw $\overline{OC}$ and note that $OA = OB = OC$. Then $\triangle OQA \cong \triangle OQB$ by SSS and hence $\angle AOQ = \angle BOQ$. If P is the point of intersection of the line $\overline{AB}$ with the line $\overline{OQ}$ (possibly extended), then $\triangle OPA \cong \triangle OPB$ by SAS and hence $\angle OPA = \angle OPB$. But these equal angles are also supplementary, so we conclude that $\angle OPA = 90^\circ$. In particular, $\angle AOP + \angle OAP = 180^\circ - \angle OPA = 90^\circ$.

Note that $\triangle AOC \cong \triangle AOB$ by SSS, since $AB = AC$, and hence $\angle OAC = \angle OAB = \angle OAP$. Also, $\triangle OQA$ is isosceles with $QO = QA$, so $\angle QAO = \angle AOQ = \angle AOP$. Thus $\angle QAC = \angle QAO + \angle OAC = \angle AOP + \angle OAP = 90^\circ$. In other words, $\overline{CA}$ is perpendicular to the radius $\overline{QA}$, and therefore $\overline{CA}$ is tangent to the circumcircle of $\triangle AOB$ at the point A.

3. Decide whether or not the number $\sqrt[3]{26-15\sqrt{3}} + \sqrt[3]{26+15\sqrt{3}}$ is an integer. Prove that your answer is correct.

SOLUTION. Let $u = \sqrt[3]{26-15\sqrt{3}}$, $v = \sqrt[3]{26+15\sqrt{3}}$, and set $x = u+v$. Then u, v and x are real numbers and our goal is to find an alternative expression for x which will more clearly indicate whether or not it is an integer. First note that $u^3+v^3 = (26-15\sqrt{3}) + (26+15\sqrt{3}) = 52$ and that $(uv)^3 = (26 - 15\sqrt{3})(26 + 15\sqrt{3}) = 26^2 - 3{\cdot}15^2 = 1$. Thus $uv = \sqrt[3]{1} = 1$. Finally, since $x = u + v$, we have

$$x^3 = (u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 = (u^3 + v^3) + 3uv(u+v) = 52 + 3x.$$

In other words, $0 = x^3 - 3x - 52 = (x-4)(x^2 + 4x + 13)$. But $x^2 + 4x + 13 = (x+2)^2 + 9 > 0$, so we must have $0 = x - 4$, and $x = 4$ is indeed an integer.

4. If n is a positive integer, let $d(n)$ denote the number of positive divisors of n. For example, $d(12) = 6$ since the six positive divisors of 12 are 1, 2, 3, 4, 6 and 12. Prove that $d(n) < 2\sqrt{n}$ for all positive integers n and find all positive integers n such that $d(n) \geq 2\sqrt{n} - 1$.

SOLUTION. Given n, let us say that positive integers a and b are *paired* if $ab = n$. (We allow $a = b$.) If a and b are paired, then both are divisors of n and if a is any divisor of n, then a is paired with exactly one number, namely $b = n/a$. Each divisor of n thus lies in exactly one pair, and so $d(n)$ is at most twice the number of pairs.

Let k be the largest integer such that $k \leq \sqrt{n}$. If a and b are paired with $a \leq b$, we have $a^2 \leq ab = n$ and so $a \leq k$. Each pair thus contains a number $\leq k$. Hence there are at most k pairs and $d(n) \leq 2k$. If $k < \sqrt{n}$, this yields $d(n) < 2\sqrt{n}$, as wanted. Otherwise, $k = \sqrt{n}$, and so the number k is paired with itself. In this case, we have $d(n) < 2k = 2\sqrt{n}$.

The numbers $n = 1$ and $n = 2$ satisfy the inequality $d(n) \geq 2\sqrt{n} - 1$, but $n = 3$ does not. To find other solutions, we can assume that $n \geq 4$, and thus $k \geq 2$. We have $d(n) \geq 2\sqrt{n} - 1 \geq 2k - 1 > 2(k-1)$, and thus the number of pairs must exceed $k - 1$. The number of pairs is therefore equal to k, and hence all of the numbers $1, 2, \ldots, k$ must be divisors of n. We have $(k+1)^2 > n$ and n must be divisible by k and $k - 1$. Hence n is divisible by $k(k-1)$, since k and $k - 1$ have no prime factors in common. But $n \geq k^2$, so we cannot have $n = k(k-1)$, and thus $n \geq 2k(k-1)$. This yields $(k+1)^2 > 2k(k-1)$, and it follows easily that $k \leq 4$.

If $k = 2$, then $4 \leq n < 9$ and n must be divisible by 2, so the possibilities for n are 4, 6 and 8. If $k = 3$, then $9 \leq n < 16$ and n must be divisible by 2 and 3, so the only possibility is $n = 12$. Finally, if $k = 4$, then $16 \leq n < 25$ and n is divisible by 3 and 4, so the only possibility here is $n = 24$. But $n = 8$ and $n = 24$ do not actually satisfy the inequality $d(n) \geq 2\sqrt{n} - 1$. Thus the full list of solutions is 1, 2, 4, 6 and 12.

5. Alice and Bob play a game by taking turns removing some stones from a pile. The rules require that the number of stones removed at each turn must be a divisor of the number of stones in the pile at the start of that turn, and no player is ever allowed to take all of the stones. The winner of the game is the last person who takes a stone. If we start with 100 stones and Alice goes first, prove that Alice can win, no matter what Bob does.

SOLUTION. Alice's plan is to always to remove just one stone. Suppose at some point, when it is Alice's turn, there are an even number of stones in the pile. (Note that this is precisely the situation at the beginning since the game starts with 100 stones.) Then Alice can remove one stone, and Bob is left with an odd number of stones. If this odd number is larger than 1, then Bob can remove some stones. But he must remove an odd number, since all divisors of an odd number are odd. Of course, by removing an odd number of stones from an odd pile, this leaves Alice again with an even sized pile. Continuing in this manner, we see that Alice can always take her single stone, but in time, Bob will be left with a pile having size one and be unable to move. Alice wins!