WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

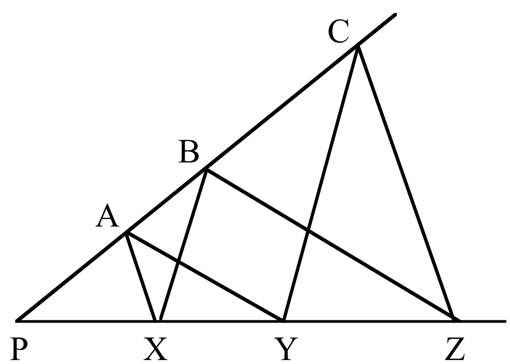
SOLUTIONS TO PROBLEM SET II (2002-2003)

1. For any positive integer n, let $S(n)$ denote the sum of its digits. Show that the equation $n + S(n) = 1{,}000{,}000$ has no solution. Then solve the equation $n + S(n) = 1{,}000{,}000{,}000$.

SOLUTION. First consider the equation $n + S(n) = 1{,}000{,}000$. Here, since $S(n) > 0$, we have $n \leq 999{,}999$. Thus n has at most 6 digits and hence $S(n) \leq 6 \cdot 9 = 54$. It follows that $n \geq 1{,}000{,}000 - 54 = 999{,}946$ and therefore $n = 999{,}9xy$ where x and y are digits (that is, integers between 0 and 9) with $x \geq 4$. Now $n = 999{,}900 + 10x + y$ and $S(n) = 36 + x + y$, so the original equation becomes $1{,}000{,}000 = n + S(n) = (999{,}900 + 10x + y) + (36 + x + y)$, and we see that $64 = 11x + 2y$. Obviously, $x \leq 5$. If $x = 5$ then $2y = 9$, and if $x = 4$ then $2y = 20$. In either case, y is not a digit, so the equation has no solution.

We study the equation $n + S(n) = 1{,}000{,}000{,}000$ in a similar manner. Here n has at most 9 digits, so $S(n) \leq 9 \cdot 9 = 81$ and $n \geq 999{,}999{,}919$. Thus $n = 999{,}999{,}9xy = 999{,}999{,}900 + 10x + y$, with $x \geq 1$, and $S(n) = 63 + x + y$. Hence the equation becomes $1{,}000{,}000{,}000 = n + S(n) = (999{,}999{,}900 + 10x + y) + (63 + x + y)$, and we see that $37 = 11x + 2y$. Obviously, $x \leq 3$ and x is not even. If $x = 1$, then $2y = 26$ and y is too large. Thus we must have $x = 3$, so $2y = 4$ and $y = 2$. In other words, $n = 999{,}999{,}932$.

2. Points P, A, B and C are collinear, as are the points P, X, Y and Z. If $\overline{AY}$ and $\overline{BZ}$ are parallel, and if $\overline{BX}$ and $\overline{CY}$ are parallel, show that the line segments $\overline{AX}$ and $\overline{CZ}$ are also parallel.



SOLUTION. Since $\overline{AY}$ and $\overline{BZ}$ are parallel, we see that $\triangle PAY$ and $\triangle PBZ$ are similar, so $PA/PY = PB/PZ$. In the same way, since $\overline{BX}$ and $\overline{CY}$ are parallel, we see that $\triangle PBX$ and $\triangle PCY$ are similar, so $PC/PB = PY/PX$. It follows that $PA/PX = (PA/PY)(PY/PX) = (PB/PZ)(PC/PB) = PC/PZ$. Therefore $\triangle PAX$ is similar to $\triangle PCZ$ and we conclude that $\overline{AX}$ and $\overline{CZ}$ are parallel.

3. Find all real numbers x that satisfy $|x| - |x-1| + 2|x-2| = 3$.

SOLUTION. If $x \geq 2$, then $|x| = x$, $|x-1| = x-1$ and $|x-2| = x-2$. Thus $3 = |x| - |x-1| + 2|x-2| = x - (x-1) + 2(x-2) = 2x - 3$, so $x = 3$. If $1 \leq x \leq 2$, then $|x| = x$, $|x-1| = x-1$ and $|x-2| = 2-x$. Thus $3 = |x| - |x-1| + 2|x-2| = x - (x-1) + 2(2-x) = 5 - 2x$, so $x = 1$. Next, if $0 \leq x \leq 1$, then $|x| = x$, $|x-1| = 1-x$ and $|x-2| = 2-x$. Thus $3 = |x| - |x-1| + 2|x-2| = x - (1-x) + 2(2-x) = 3$, and hence this equation is satisfied by all real numbers x in the interval. Finally, if $x \leq 0$, then $|x| = -x$, $|x-1| = 1-x$ and $|x-2| = 2-x$. Thus $3 = |x| - |x-1| + 2|x-2| = -x - (1-x) + 2(2-x) = 3 - 2x$, so $x = 0$. We conclude that the equation $|x| - |x-1| + 2|x-2| = 3$ is satisfied by $x = 3$ and by all x with $0 \leq x \leq 1$.

4. Let $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ and in general, $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. (This is the famous *Fibonacci sequence*.) Show that F_{2n}/F_n is always an integer.

SOLUTION. For convenience, define $F_0 = 0$ and note that $F_2 = F_1 + F_0$, so $F_i = F_{i-1} + F_{i-2}$ holds for all $i \geq 2$. Fix a positive integer n and write $G_i = F_{n+i} - F_i \cdot F_{n-1}$ for all $i \geq 0$. Then $G_0 = F_n - 0 \cdot F_{n-1} = F_n$ and $G_1 = F_{n+1} - 1 \cdot F_{n-1} = F_n$, since $F_{n+1} = F_n + F_{n-1}$. Furthermore, for all $i \geq 2$, we have

$$\begin{aligned} G_i = F_{n+i} - F_i \cdot F_{n-1} &= \big(F_{n+i-1} + F_{n+i-2}\big) - \big(F_{i-1} + F_{i-2}\big) \cdot F_{n-1} \\ &= \big(F_{n+i-1} - F_{i-1} \cdot F_{n-1}\big) + \big(F_{n+i-2} - F_{i-2} \cdot F_{n-1}\big) = G_{i-1} + G_{i-2}. \end{aligned}$$

We use mathematical induction to show that all G_i are divisible by F_n, and this is certainly true for $i = 0$ and $i = 1$. Now suppose that $i \geq 2$ and that we already know this divisiblity holds for all subscripts $j \leq i - 1$. Then F_n divides G_{i-1} and G_{i-2}, and therefore F_n divides $G_{i-1} + G_{i-2} = G_i$. By mathematical induction, we now know that F_n divides all G_i. Finally, set $i = n$. Then F_n divides $G_n = F_{2n} - F_n \cdot F_{n-1}$, and hence F_n divides $G_n + F_n \cdot F_{n-1} = F_{2n}$.

5. Let a, b and c be fixed nonnegative integers, and assume that $n^3 + an^2 + bn + c$ is a perfect cube for infinitely many nonnegative integers n. Show that $n^3 + an^2 + bn + c$ is a perfect cube for all integers n.

SOLUTION. Choose a positive integer t so that $3t \geq a$, $3t^2 \geq b$ and $t^3 \geq c$. Then for any integer $n \geq 0$, we have

$$n^3 \leq n^3 + an^2 + bn + c \leq n^3 + 3tn^2 + 3t^2n + t^3 = (n + t)^3.$$

In particular, if $n^3 + an^2 + bn + c$ is a perfect cube, then we must have $n^3 + an^2 + bn + c = (n + k)^3$ for some integer k with $0 \leq k \leq t$. Now there are only finitely many choices for k and yet we are told that there are infinitely many integers $n \geq 0$ with $n^3 + an^2 + bn + c$ a perfect cube. Thus, for some fixed k, the equation $n^3 + an^2 + bn + c = (n + k)^3$ must hold for infinitely many distinct integers n. In particular, the polynomial $X^3 + aX^2 + bX + c - (X + k)^3 = (a - 3k)X^2 + (b - 3k^2)X + (c - k^3)$ has infinitely many roots. Since any nonzero polynomial can have only finitely many distinct roots, we conclude that the above polynomial must have all coefficients equal to 0, and hence $a = 3k$, $b = 3k^2$ and $c = k^3$. Thus, $n^3 + an^2 + bn + c = n^3 + 3kn^2 + 3k^2n + k^3 = (n + k)^3$ is always a perfect cube.