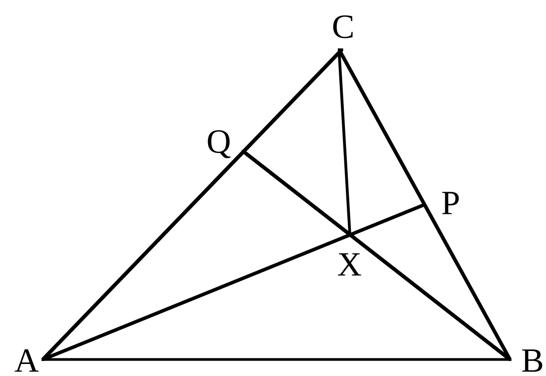
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2002-2003)

1. In how many different ways can \$100.00 be made from 5-cent, 10-cent and 25-cent coins if it is required that exactly 1000 coins are used?

SOLUTION. Suppose we use n of the 5-cent coins, d of the 10-cent coins, and q of the 25-cent coins. Since we are told that 1000 coins are used, we have $(*)\quad n + d + q = 1000$. Futhermore, since \$100.00 is equal to $10,000$ cents, it follows that $5n + 10d + 25q = 10000$ and hence $(**)\quad n+2d+5q = 2000$. Subtracting $(*)$ from $(**)$ yields $d+4q = 1000$, so $d = 1000-4q$. Furthermore, by subtracting $(*)$ from 2 times equation $(**)$, we obtain $n - 3q = 0$, so $n = 3q$. Conversely, if $d = 1000 - 4q$ and if $n = 3q$, then equations $(*)$ and $(**)$ are easily seen to be satisfied. Finally, since n, d, and q must be nonnegative integers, we have $q = 0, 1, 2, \dots, 250$, and there are 251 possible solutions.

2. In triangle ABC, let P be the point on side $\overline{CB}$ satisfying $CP = (1/2)CB$ and let Q be the point on side $\overline{CA}$ with $CQ = (1/3)CA$. Draw the lines $\overline{AP}$ and $\overline{BQ}$, as indicated and suppose these lines meet at the point X. Find the ratio of the area of $\triangle ABX$ to the area of $\triangle ABC$.



SOLUTION. We can view $\overline{BC}$ as the base of $\triangle ABC$ and $\overline{BP}$ as the base of $\triangle ABP$. These two triangles thus have equal heights and the base of $\triangle ABP$ is half that of $\triangle ABC$. It follows from this that the area of $\triangle ABP$ is one half that of $\triangle ABC$. By similar reasoning, the area of $\triangle BCQ$ is one third of the area of $\triangle ABC$.

We introduce the notation $[ABC]$ to denote the area of $\triangle ABC$ and similarly for other triangles. Thus $[ABP] = \frac{1}{2}[ABC]$ and $[BCQ] = \frac{1}{3}[ABC]$. In order to avoid fractions, it is convenient to choose units so that $[ABC] = 6$, in which case we have $[ABP] = 3 = [ACP]$ and $[BCQ] = 2$ and $[ACQ] = 4$.

Now draw $\overline{CX}$ and let $x = [AXB]$, so that $[BXP] = 3 - x$. We see that $[BXP] = [CXP]$ since $\triangle BXP$ and $\triangle CXP$ have equal base lengths BP and PC and equal heights. Thus $[CXP] = 3 - x$ and hence $[BXC] = 2(3 - x)$. Next, we observe that

$$[CQX] = [BCQ] - [BCX] = 2 - 2(3 - x) = 2x - 4.$$

Furthermore, since $AQ = 2CQ$, it follows that

$$[AQX] = 2[CQX] = 2(2x - 4) = 4x - 8.$$

But $[AQX] + [ABX] = [AQB] = 4$, so we get the equation $(4x - 8) + x = 4$. Thus $5x = 12$ and $x = 12/5$. We conclude that $[ABX]/[ABC] = (12/5)/6 = 2/5$.

3. Let x and y be nonzero real numbers satisfying the equation $x + 6/x = 2y + 3/y$. If $x/y \neq 2$, find the product xy.

SOLUTION. By assumption, we have

$$\begin{aligned} 0 &= \big(x + 6/x\big) - \big(2y + 3/y\big) = \big(x - 2y\big) - 3\,\big(1/y - 2/x\big) \\ &= \big(x - 2y\big) - 3\,\big(x - 2y\big)/(xy) = \big(x - 2y\big)\,\big(1 - 3/(xy)\big). \end{aligned}$$

If $x - 2y = 0$, then $x = 2y$ and $x/y = 2$, which we are told is not the case. Thus we must have $1 - 3/(xy) = 0$ and $xy = 3$.

For an alternative solution, write $a = x + 6/x = 2y + 3/y$. Since $x + 6/x = a$ and $x(6/x) = 6$, we see that $\{x, 6/x\}$ is the set of solutions to the quadratic equation in T given by

$$(T - x)(T - 6/x) = T^2 - aT + 6 = 0.$$

Similarly, $2y + 3/y = a$ and $(2y)(3/y) = 6$, so $\{2y, 3/y\}$ is also the set of solutions to the equation $T^2 - aT + 6 = 0$. Thus $\{x, 6/x\} = \{2y, 3/y\}$ and we see that either $x = 2y$ or $x = 3/y$. In particular, either $x/y = 2$ or $xy = 3$.

4. Let $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, and in general, $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. (This is the famous *Fibonacci sequence*.) Show that $F_n/F_{n-1} < 1.7$ for all $n \geq 4$.

SOLUTION. We use mathematical induction on n to show that $F_n/F_{n-1} < 1.7$ for all $n \geq 4$. To start with, we have $F_4/F_3 = 3/2 = 1.5 < 1.7$ and $F_5/F_4 = 5/3 < 1.67 < 1.7$, and therefore we need only consider cases where $n \geq 6$. Let us assume that we have already proved the inequalities $F_k/F_{k-1} < 1.7$ when $4 \leq k \leq n-1$, and so in particular, we have $F_{n-1}/F_{n-2} < 1.7$ and $F_{n-2}/F_{n-3} < 1.7$. (These inequalities are obtained by taking $k = n-1$ and $k = n-2$, and they are valid since $4 \leq k \leq n-1$ in both cases.) Thus, $F_{n-1} < (1.7)F_{n-2}$ and $F_{n-2} < (1.7)F_{n-3}$, and therefore we have

$$F_n = F_{n-1} + F_{n-2} < (1.7)F_{n-2} + (1.7)F_{n-3} = (1.7)(F_{n-2} + F_{n-3}) = (1.7)F_{n-1}\,.$$

We conclude that $F_n/F_{n-1} < 1.7$, and the result follows by induction.

5. Find all positive integers n such that $n^2 + 25\,n + 19$ is a perfect square.

SOLUTION. Suppose $n^2 + 25\,n + 19$ is a perfect square. Since

$$(n+4)^2 = n^2 + 8\,n + 16 < n^2 + 25\,n + 19 \quad \text{and}$$

$$(n+13)^2 = n^2 + 26\,n + 169 > n^2 + 25\,n + 19,$$

it follows that

$$n^2 + 25\,n + 19 = (n+k)^2 = n^2 + 2nk + k^2$$

for some integer k with $5 \leq k \leq 12$. Thus, by subtracting n^2 from both sides above, we have $(\dagger)\quad n(25 - 2k) = k^2 - 19$ and, in particular, $25 - 2k$ must divide $k^2 - 19$. By considering the eight possible integers k between 5 and 12, we see that k can equal only 8, 11 or 12. When $k = 8$, equation $(\dagger)$ becomes $n \cdot 9 = 45$, so $n = 5$. When $k = 11$, we have $n \cdot 3 = 102$, so $n = 34$. Finally, when $k = 12$, we get $n \cdot 1 = 125$ and $n = 125$. Since $n^2 + 25\,n + 19 = (n+k)^2$ for all such pairs (n, k), we conclude that $n = 5$, 34 and 125 are the only possibilities.