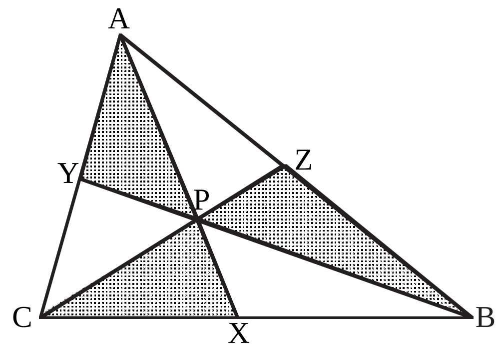
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

PROBLEM SET III (2001-2002) **DECEMBER 2001**



1. Find all pairs of positive integers x and y that satisfy $5x - 7y = 1$ and $x/y > \sqrt{2}$.

2. Point P lies in the interior of $\triangle ABC$. Lines $\overline{AX}$, $\overline{BY}$ and $\overline{CZ}$ are drawn, all of them going through P, as shown. These lines divide the interior of the triangle into six smaller triangles. Show that if the three shaded triangles have equal areas, then all six small triangles have equal areas.

3. Given a positive odd integer m, we define $m^* = (3m+1)/2^a$, where a is chosen so that m^* is an odd integer. For example, if $m = 7$, then $3m + 1 = 22$, so $2^a = 2$ and $7^* = 11$. The $*$-sequence beginning with m is the sequence of odd numbers obtained from m by repeatedly applying $*$. For example, the $*$-sequence starting with 7 is $7, 11, 17, 13, 5, 1$, where we stopped with 1 because $1^* = 1$. (It is believed that *every* $*$-sequence reaches the number 1, but no one has been able to prove this.) Show that if the odd number m exceeds 1, then the $*$-sequence beginning with m must contain two numbers n and n^* such that $n > n^*$.

4. Your calculator will tell you that the number $\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}$ is very nearly an integer. Decide whether or not it is *exactly* an integer.

5. Let S be a finite set and suppose that $\mathcal{A}$ is a nonempty collection of subsets of S. If each of the sets in $\mathcal{A}$ contains more than half of the elements of S, show that there is some element of S that is in more than half of the members of $\mathcal{A}$.

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on the problem set page. Remember that solutions usually require a proof or justification.

RETURN TO:	MATHEMATICS TALENT SEARCH	DEADLINE
	Dept. of Mathematics, 480 Lincoln Drive	January 7
	University of Wisconsin, Madison, WI 53706	2002

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(Please Detach)

Last Name First Name Grade

School Town

Home Address Town Zip Code

PROBLEM	SCORE
1	
2	
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5	

PROBLEM SET III