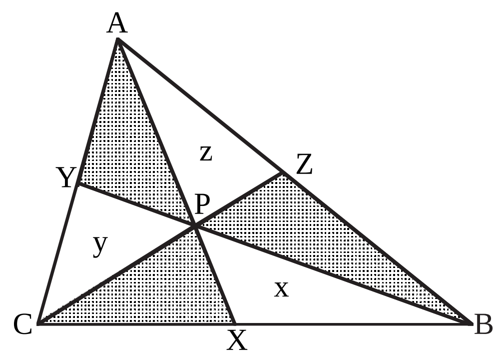
# WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

## SOLUTIONS TO PROBLEM SET III (2001-2002)

**1.** Find all pairs of positive integers $x$ and $y$ that satisfy $5x - 7y = 1$ and $x/y > \sqrt{2}$.

**SOLUTION.** Note that $(x, y) = (3, 2)$ is a solution to $5x-7y = 1$. Thus, for any integer solution $(x, y)$ to this equation, we have $5x - 7y = 1 = 5{\cdot}3 - 7{\cdot}2$, and therefore $5(x - 3) = 7(y - 2)$. But 5 and 7 are distinct prime numbers, so the latter formula implies that $x - 3 = 7a$ and $y - 2 = 5a$ for some integer $a$. In other words, $x = 7a + 3$ and $y = 5a + 2$. In addition, we are given that $x/y > \sqrt{2}$, so $x^2 > 2y^2$ and hence $(7a + 3)^2 > 2(5a + 2)^2$. This simplifies to $0 > a^2 - 2a - 1 = (a - 1)^2 - 2$, so $2 > (a - 1)^2$ and $a = 0,\ 1$ or 2. Therefore we have only three possibilities, namely $(x, y) = (3, 2)$, $(10, 7)$ and $(17, 12)$.

**2.** Point $P$ lies in the interior of $\triangle ABC$. Lines $\overline{AX}$, $\overline{BY}$ and $\overline{CZ}$ are drawn, all of them going through $P$, as shown. These lines divide the interior of the triangle into six smaller triangles. Show that if the three shaded triangles have equal areas, then all six small triangles have equal areas.




**SOLUTION.** By choosing our units appropriately, we can assume that the area of each of the three shaded triangles is 1 square unit and we let $x$, $y$ and $z$ be the areas of $\triangle BXP$, $\triangle CYP$ and $\triangle AZP$, as shown. Since $\triangle PBX$ and $\triangle PCX$ have equal heights, it follows that the ratio of their areas is equal to the ratio of their bases, and thus we have $x/1 = BX/CX$. Similarly, since the heights of $\triangle ABX$ and $\triangle ACX$ are equal, we have $(1+x+z)/(2+y) = BX/CX$. But we have already seen that $x = BX/CX$, so we obtain $x = (1 + x + z)/(2 + y)$ and therefore $x(2 + y) = 1 + x + z$. Subtracting $x$ from both sides yields $x(1 + y) = 1 + z$, and thus $x = (1 + z)/(1 + y)$. Similar reasoning also yields the equations $y = (1 + x)/(1 + z)$ and $z = (1 + y)/(1 + x)$.

The three equations we have just obtained are exactly the ones contained in Problem 3 of the previous problem set. Since we certainly know that $x$, $y$ and $z$ are positive, the result of that previous problem tells us that the only possibile solution is $x = y = z = 1$, and thus all six small triangles have equal areas.

**3.** Given a positive odd integer $m$, we define $m^* = (3m + 1)/2^a$, where $a$ is chosen so that $m^*$ is an odd integer. For example, if $m = 7$, then $3m + 1 = 22$, so $2^a = 2$ and $7^* = 11$. The $*$-sequence beginning with $m$ is the sequence of odd numbers obtained from $m$ by repeatedly applying $*$. For example, the $*$-sequence starting with 7 is 7, 11, 17, 13, 5, 1, where we stopped with 1 because $1^* = 1$. (It is believed that *every* $*$-sequence reaches the number 1, but no one has been able to prove this.) Show that if the odd number $m$ exceeds 1, then the $*$-sequence beginning with $m$ must contain two numbers $n$ and $n^*$ such that $n > n^*$.

**SOLUTION.** Every odd number $n \geq 1$ can be written as $n = 2k - 1$ for some integer $k \geq 1$. In fact, if $2^b$ is the largest power of 2 dividing $2k$, then $n = 2^b c - 1$ where $c$ is odd and $b \geq 1$. Let us call $b$ the *exponent* of $n$, and note that $3n+1 = 3(2^b c - 1) + 1 = 2^b(3c) - 2$. In particular, $(3n+1)/2 = 2^{b-1}(3c) - 1$. If $b \geq 2$, then this number is odd since $b - 1 \geq 1$, and it follows that $n^* = (3n+1)/2 > n$ and that $n^* = 2^{b-1}(3c) - 1$ has exponent $b-1$. In other words, if the exponent of $n$ is strictly larger than 1, then $n^* > n$ while the exponent of $n^*$ is 1 smaller than that of $n$. On the other hand, if $n > 1$ has exponent 1, then $n = 2c - 1$ with $c$ odd, so $(3n+1)/2 = 3c - 1$ is even. Thus $3n+1$ is divisible by 4, and therefore $n^* \leq (3n+1)/4 < n$ since $n > 1$.

Now we start with $m > 1$ and suppose the exponent of $m$ is equal to $a \geq 1$. If we write $m_0 = m$, $m_1 = m_0^*$, $m_2 = m_1^*$, and so on, then the above comments imply that $m_1$ has exponent $a-1$, $m_2$ have exponent $a-2$, and the exponents continues to lower by 1 until we reach $m_{a-1}$ which has exponent equal to $a - (a-1) = 1$. Thus $1 < m_0 < m_1 < \cdots < m_{a-1}$ and again the above comments yield $m_a = m_{a-1}^* < m_{a-1}$. In other words, we have found a number $n = m_{a-1}$ in the $*$-sequence beginning with $m$ such that $n > n^*$.

**4.** Your calculator will tell you that the number $\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}$ is very nearly an integer. Decide whether or not it is *exactly* an integer.

**SOLUTION.** For simplicity, let us write $a = \sqrt[3]{\sqrt{5}+2}$ and $b = \sqrt[3]{\sqrt{5}-2}$. Our goal is to find $x = a - b$. Now we have

$$x^3 = (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a-b) = a^3 - b^3 - 3abx.$$

Thus, since $a^3 - b^3 = (\sqrt{5}+2) - (\sqrt{5}-2) = 4$ and $ab = \sqrt[3]{(\sqrt{5}+2)(\sqrt{5}-2)} = \sqrt[3]{1} = 1$, we see that $x^3 = 4 - 3x$. This equation has $x = 1$ as a root. Indeed, $x^3 + 3x - 4 = (x-1)(x^2+x+4)$ and, since the quadratic equation $x^2 + x + 4 = 0$ has no real roots, we must have $x = 1$.

**5.** Let $S$ be a finite set and suppose that $\mathcal{A}$ is a nonempty collection of subsets of $S$. If each of the sets in $\mathcal{A}$ contains more than half of the elements of $S$, show that there is some element of $S$ that is in more than half of the members of $\mathcal{A}$.

**SOLUTION.** Since each member of $\mathcal{A}$ contains more than half the elements of $S$, it is clear that $S$ is not the empty set. Let $a > 0$ be the number of members in the nonempty collection $\mathcal{A}$ and let $s > 0$ be the number of elements of $S$. We know that each member of $\mathcal{A}$ contains more than $s/2$ elements. In particular, if we let $t$ denote the total of the sizes of all of the members of $\mathcal{A}$, then it follows that $t > a(s/2)$.

For each element $x$ of $S$, let us use $n_x$ to denote the number of members of $\mathcal{A}$ that contain $x$. Our goal is to show that $n_y > a/2$ for some element $y$ in $S$. Now, if we sum the numbers $n_x$ for all $x$ in $S$, then we see that we are counting the members of $\mathcal{A}$ repeatedly. In fact, each member $A$ in $\mathcal{A}$ is counted as many times as there are elements in $A$. In other words, the sum of the numbers $n_x$ is equal to the same number $t$ we considered before, namely the total of the sizes of the members of $\mathcal{A}$. The average of the $s$ numbers $n_x$ is therefore equal to $t/s$. Of course, it is not possible for all the numbers $n_x$ to be below average. Therefore $n_y \geq t/s$ for some element $y$ in $S$. But we know that $t > as/2$, and thus $n_y \geq t/s > a/2$, precisely what we wanted to prove.