1. Find all pairs of positive integers $x$ and $y$ that satisfy $5x - 7y = 1$ and $x/y > \sqrt{2}$.

**SOLUTION.** Note that $(x, y) = (3, 2)$ is a solution to $5x - 7y = 1$. Thus, for any integer solution $(x, y)$ to this equation, we have $5x - 7y = 1 = 5.3 - 7.2$, and therefore $5(x - 3) = 7(y - 2)$. But 5 and 7 are distinct prime numbers, so the latter formula implies that $x - 3 = 7a$ and $y - 2 = 5a$ for some integer $a$. In other words, $x = 7a + 3$ and $y = 5a + 2$. In addition, we are given that $x/y > \sqrt{2}$, so $x^2 > 2y^2$ and hence $(7a + 3)^2 > 2(5a + 2)^2$. This simplifies to $0 > a^2 - 2a - 1 = (a - 1)^2 - 2$, so $2 > (a - 1)^2$ and $a = 0$, 1 or 2. Therefore we have only three possibilities, namely $(x, y) = (3, 2), (10, 7)$ and $(17, 12)$.

2. Point $P$ lies in the interior of $\triangle ABC$. Lines $AX, BY$ and $CZ$ are drawn, all of them going through $P$, as shown. These lines divide the interior of the triangle into six smaller triangles. Show that if the three shaded triangles have equal areas, then all six small triangles have equal areas.

**SOLUTION.** By choosing our units appropriately, we can assume that the area of each of the three shaded triangles is 1 square unit and we let $x$, $y$ and $z$ be the areas of $\triangle BX P$, $\triangle CY P$ and $\triangle AZ P$, as shown. Since $\triangle P B X$ and $\triangle P C X$ have equal heights, it follows that the ratio of their areas is equal to the ratio of their bases, and thus we have $x/1 = BX/CX$. Similarly, since the heights of $\triangle ABX$ and $\triangle ACX$ are equal, we have $(1 + x + z)/(2 + y) = BX/CX$. But we have already seen that $x = BX/CX$, so we obtain $x = (1 + x + z)/(2 + y)$ and therefore $x(2 + y) = 1 + x + z$. Subtracting $x$ from both sides yields $x(1 + y) = 1 + z$, and thus $x = (1 + z)/(1 + y)$. Similar reasoning also yields the equations $y = (1 + x)/(1 + z)$ and $z = (1 + y)/(1 + x)$.

The three equations we have just obtained are exactly the ones contained in Problem 3 of the previous problem set. Since we certainly know that $x$, $y$ and $z$ are positive, the result of that previous problem tells us that the only possible solution is $x = y = z = 1$, and thus all six small triangles have equal areas.

3. Given a positive odd integer $m$, we define $m^* = (3m + 1)/2^a$, where $a$ is chosen so that $m^*$ is an odd integer. For example, if $m = 7$, then $3m + 1 = 22$, so $2^a = 2$ and $7^* = 11$. The $*$-sequence beginning with $m$ is the sequence of odd numbers obtained from $m$ by repeatedly applying $*$. For example, the $*$-sequence starting with 7 is 7, 11, 17, 13, 5, 1, where we stopped with 1 because $1^* = 1$. (It is believed that every $*$-sequence reaches the number 1, but no one has been able to prove this.) Show that if the odd number $m$ exceeds 1, then the $*$-sequence beginning with $m$ must contain two numbers $n$ and $n^*$ such that $n > n^*$.

**SOLUTION.** Every odd number $n \geq 1$ can be written as $n = 2k - 1$ for some integer $k \geq 1$. In fact, if $2^b$ is the largest power of 2 dividing $2k$, then $n = 2^bc - 1$ where $c$ is odd and $b \geq 1$. Let
us call \(b\) the exponent of \(n\), and note that \(3n + 1 = 3(2^b c - 1) + 1 = 2^b (3c) - 2\). In particular, \((3n + 1)/2 = 2^{b-1}(3c) - 1\). If \(b \geq 2\), then this number is odd since \(b - 1 \geq 1\), and it follows that \(n^* = (3n + 1)/2 > n\) and that \(n^* = 2^{b-1}(3c) - 1\) has exponent \(b - 1\). In other words, if the exponent of \(n\) is strictly larger than 1, then \(n^* > n\) while the exponent of \(n^*\) is 1 smaller than that of \(n\). On the other hand, if \(n > 1\) has exponent 1, then \(n = 2c - 1\) with \(c\) odd, so \((3n + 1)/2 = 3c - 1\) is even. Thus \(3n + 1\) is divisible by 4, and therefore \(n^* \leq (3n + 1)/4 < n\) since \(n > 1\).

Now we start with \(m > 1\) and suppose the exponent of \(m\) is equal to \(a \geq 1\). If we write \(m_0 = m, m_1 = m_0^*, m_2 = m_1^*, \) and so on, then the above comments imply that \(m_1\) has exponent \(a - 1\), \(m_2\) have exponent \(a - 2\), and the exponents continues to lower by 1 until we reach \(m_{a-1}\) which has exponent equal to \(a - (a - 1) = 1\). Thus \(1 < m_0 < m_1 < \cdots < m_{a-1}\) and again the above comments yield \(m_a = m_{a-1}^* < m_{a-1}\). In other words, we have found a number \(n = m_{a-1}\) in the \(*\)-sequence beginning with \(m\) such that \(n > n^*\).

4. Your calculator will tell you that the number \(\sqrt[3]{5 + 2} - \sqrt[3]{5 - 2}\) is very nearly an integer. Decide whether or not it is exactly an integer.

**SOLUTION.** For simplicity, let us write \(a = \sqrt[3]{5 + 2}\) and \(b = \sqrt[3]{5 - 2}\). Our goal is to find \(x = a - b\). Now we have

\[
x^3 = (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b) = a^3 - b^3 - 3abx.
\]

Thus, since \(a^3 - b^3 = (\sqrt[3]{5} + 2) - (\sqrt[3]{5} - 2) = 4\) and \(ab = \sqrt[3]{(\sqrt[3]{5} + 2)(\sqrt[3]{5} - 2)} = \sqrt[3]{1} = 1\), we see that \(x^3 = 4 - 3x\). This equation has \(x = 1\) as a root. Indeed, \(x^3 + 3x - 4 = (x - 1)(x^2 + x + 4)\) and, since the quadratic equation \(x^2 + x + 4 = 0\) has no real roots, we must have \(x = 1\).

5. Let \(S\) be a finite set and suppose that \(A\) is a nonempty collection of subsets of \(S\). If each of the sets in \(A\) contains more than half of the elements of \(S\), show that there is some element of \(S\) that is in more than half of the elements of \(A\).

**SOLUTION.** Since each member of \(A\) contains more than half the elements of \(S\), it is clear that \(S\) is not the empty set. Let \(a > 0\) be the number of members in the nonempty collection \(A\) and let \(s > 0\) be the number of elements of \(S\). We know that each member of \(A\) contains more than \(s/2\) elements. In particular, if we let \(t\) denote the total of the sizes of all of the members of \(A\), then it follows that \(t > a(s/2)\).

For each element \(x\) of \(S\), let us use \(n_x\) to denote the number of members of \(A\) that contain \(x\). Our goal is to show that \(n_y > a/2\) for some element \(y\) in \(S\). Now, if we sum the numbers \(n_x\) for all \(x\) in \(S\), then we see that we are counting the members of \(A\) repeatedly. In fact, each member \(A\) in \(A\) is counted as many times as there are elements in \(A\). In other words, the sum of the numbers \(n_x\) is equal to the same number \(t\) we considered before, namely the total of the sizes of the members of \(A\). The average of the \(s\) numbers \(n_x\) is therefore equal to \(t/s\). Of course, it is not possible for all the numbers \(n_x\) to be below average. Therefore \(n_y \geq t/s\) for some element \(y\) in \(S\). But we know that \(t > as/2\), and thus \(n_y \geq t/s > a/2\), precisely what we wanted to prove.