1. For how many integers \( n \) is the quantity \( n^2 - 20n + 244 \) equal to a perfect square?

2. In the figure, \( \triangle ABC \) is equilateral and \( P \) is some point in the interior of the triangle. Perpendiculars \( PR \), \( PS \) and \( PT \) are dropped from \( P \) to the sides of the triangle, and lines are drawn from \( P \) to the vertices \( A \), \( B \) and \( C \). Show that the sum of the areas of the three shaded triangles is exactly half of the area of \( \triangle ABC \).

3. Find all positive real numbers \( x \), \( y \) and \( z \) such that
   \[
   x = \frac{1 + z}{1 + y}, \quad y = \frac{1 + x}{1 + z}, \quad z = \frac{1 + y}{1 + x}.
   \]

4. Find a positive integer \( n \) such that the following is necessarily true: Suppose I have \( n^2 \) stones, each of which is either red, white, blue or green, and suppose that I place one of these stones at the center of each of the \( n^2 \) boxes of an \( n \times n \) square grid. Then there must exist a stone such that both its row and column contain another stone of the same color.

5. Let \( S \) be a finite set and recall that two subsets \( X \) and \( Y \) of \( S \) are said to be disjoint if they have no elements in common. Suppose that a collection \( A \) of subsets of \( S \) has the property that no two of the sets in \( A \) are disjoint but that every subset of \( S \) that is not in \( A \) is disjoint from some member of \( A \). Prove that \( A \) contains exactly half of the subsets of \( S \).

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on the problem set page. Remember that solutions usually require a proof or justification.