WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2001-2002)

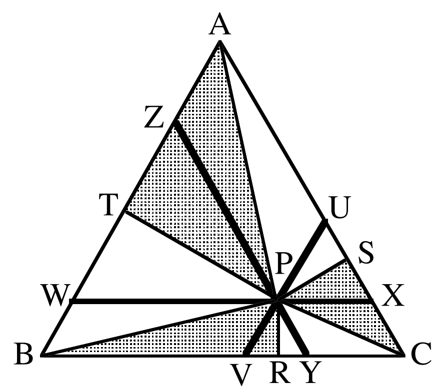
1. For how many integers n is the quantity $n^2 - 20n + 244$ equal to a perfect square?

SOLUTION. Since $n^2 - 20n + 244 = (n-10)^2 + 144$, it is convenient to write $m = n - 10$. We thus need to find the number of integers m such that $m^2 + 144$ is a perfect square. Suppose that $m^2 + 144 = q^2$, where q is positive. Then $144 = q^2 - m^2 = (q-m)(q+m)$, and this yields a factorization of the number 144 into two integers. Also, $q > m$, and thus both factors are positive.

Suppose that we have any such factorization: $144 = ab$. We want to solve $q - m = a$ and $q + m = b$, and from this, we see that $m = (b-a)/2$ and $q = (b+a)/2$. Since we require that m should be an integer, we see that either a and b are both even or else they are both odd. The latter alternative is impossible, however, since $ab = 144$ is even. We must therefore count the even positive divisors a of 144 such that $b = 144/a$ is also even.

If we write $a = 2u$ and $b = 2v$, then we have $144 = 4uv$. Thus $36 = uv$, and u is some divisor of $36 = 2^2 3^2$. We see that 36 has exactly nine positive divisors and it follows that there are exactly nine possibilities for a. Conversely, if a is any even positive divisor of 144 such that $b = 144/a$ is also even, we can let $m = (b-a)/2$ and $q = (b+a)/2$ so that $q + m = b$ and $q - m = a$. Then $144 = ab = (q-m)(q+m) = q^2 - m^2$, so $m^2 + 144 = q^2$ is a square, as required. It follows that there are exactly nine values of m for which $m^2 + 144$ is square, and thus there are nine solutions to the original problem, namely: $n = -25, -6, 1, 5, 10, 15, 19, 26$ and 45.

2. In the figure, $\triangle ABC$ is equilateral and P is some point in the interior of the triangle. Perpendiculars $\overline{PR}$, $\overline{PS}$ and $\overline{PT}$ are dropped from P to the sides of the triangle, and lines are drawn from P to the vertices A, B and C. Show that the sum of the areas of the three shaded triangles is exactly half of the area of $\triangle ABC$.



SOLUTION. Draw lines $\overline{UV}$, $\overline{WX}$ and $\overline{YZ}$ parallel to the sides of the given equilateral triangle, as shown. Note that each angle of $\triangle PVY$ is 60°, and thus this triangle is equilateral. It follows that the altitude $\overline{PR}$ of this triangle divides it into two congruent triangles, one of which is shaded. We see, therefore, that exactly half of the area of equilateral $\triangle PVY$ is shaded, and similarly, exactly half of the area of $\triangle PXU$ is shaded and exactly half of the area of $\triangle PZW$ is shaded.

Now consider parallelogram $AZPU$. Its diagonal AP divides it into two congruent triangles, one of which is shaded. Thus exactly half of the area of $AZPU$ is shaded and the same is true for parallelogram $BVPW$ and for parallelogram $CXPY$.

We now see that $\triangle ABC$ is divided into three equilateral triangles and three parallelograms, and each of these six pieces has exactly half of its area shaded. It follows that exactly half of the total area is shaded, and this is what we wanted to prove.

3. Find all positive real numbers x, y and z such that

$$x = \frac{1+z}{1+y} \qquad y = \frac{1+x}{1+z} \qquad z = \frac{1+y}{1+x}.$$

SOLUTION. Suppose that x, y and z are positive and satisfy the three given equations. If we multiply these equations, we see that $xyz = 1$, and thus the largest of x, y and z must be at least 1. We will prove that $x = 1$, $y = 1$ and $z = 1$.

We can assume that z is the largest of the unknowns, so that $z \geq 1$ and $z \geq y$. Now $1 + z \geq 1 + y$, and since these are positive quantities, we see that $x = (1 + z)/(1 + y) \geq 1$. We now know that $z \geq 1$ and $x \geq 1$, and so $y = 1/(xz) \leq 1 \leq x$. Thus $1 + y \leq 1 + x$, and since these are positive quantities, we conclude that $z = (1 + y)/(1 + x) \leq 1$. But we already know that $z \geq 1$, so $z = 1$. Therefore, $1 = z = (1 + y)/(1 + x)$, and hence $y = x$. Since $1 = xyz = x^2$, we conclude that $x = 1$ and $y = 1$.

4. Find a positive integer n such that the following is necessarily true: Suppose I have n^2 stones, each of which is either red, white, blue or green, and suppose that I place one of these stones at the center of each of the n^2 boxes of an $n \times n$ square grid. Then there must exist a stone such that both its row and column contain another stone of the same color.

SOLUTION. This is a continuation of last month's problem #4 which considered 3 colors on a 4×4 grid. In the case of 4 colors, we show that $n = 13$ works, although it is certainly possible that a smaller value of n will suffice. Let $n = 13$. Since the fifth row of the 13×13 grid contains 13 stones of four different colors, one of these colors must occur at least four times. By symmetry, we can assume that there are four green stones and, for convenience, we can assume that they appear in the first four columns. If another green stone appears in the first column, then the stone in row 5 and column 1 has another stone of the same color in its row and column. Thus, we can assume that there are no other green stones in the first column, and similarly no others in the second, third and fourth columns. It follows that the 4×4 grid determined by the first four rows and the first four columns of our 13×13 grid contains only red, white and blue stones. By the result of last month's problem #4, there is a stone in this 4×4 grid such that both its row and its column contain another stone of the same color.

5. Let S be a finite set and recall that two subsets X and Y of S are said to be *disjoint* if they have no elements in common. Suppose that a collection $\mathcal{A}$ of subsets of S has the property that no two of the sets in $\mathcal{A}$ are disjoint but that every subset of S that is not in $\mathcal{A}$ is disjoint from some member of $\mathcal{A}$. Prove that $\mathcal{A}$ contains exactly half of the subsets of S.

SOLUTION. If X is a subset of S, we write X^c to denote the complement of X in S, so that X^c is the set of all elements of S that are not in X. Note that $(X^c)^c = X$, and thus we can think of the subsets of S as coming in pairs: each subset X is paired with its complement X^c. The number of pairs, therefore, is exactly half of the total number of subsets. To complete the proof, we will show that the collection $\mathcal{A}$ consists of exactly one set from each such pair.

No two members of $\mathcal{A}$ are disjoint, and thus $\mathcal{A}$ cannot contain both a subset X and its complement X^c. What remains is to show that given any subset X of S, either X is a member of $\mathcal{A}$ or else X^c is a member of $\mathcal{A}$. If X is not in $\mathcal{A}$, then by assumption, there is a member Y of $\mathcal{A}$ disjoint from X, and thus Y is contained in X^c. Since Y is not disjoint from any member of $\mathcal{A}$, it follows that the set X^c, which contains Y, cannot be disjoint from any member of $\mathcal{A}$. We conclude from the hypothesis that X^c must be a member of $\mathcal{A}$, and the proof is complete.