WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

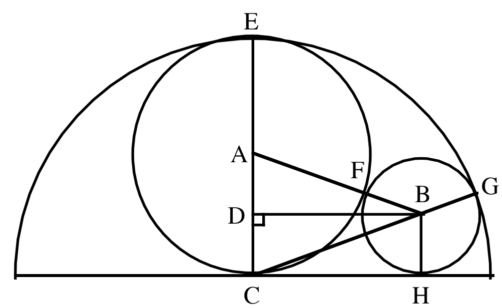
SOLUTIONS TO PROBLEM SET I (2001-2002)

1. For how many positive integers x does there exist a positive integer y with $xy/(x+y) = 100$?

SOLUTION. Let n be a fixed positive integer. We consider the more general problem of finding the number of positive integers x for which there exists a positive integer y with $xy/(x+y) = n$. Now the preceding equation yields $xy = n(x+y)$ and hence $y = nx/(x-n)$. In particular, y is a positive integer if and only if $x - n$ is a positive integer that divides nx. But $nx = n^2 + n(x-n)$, so we see that $x - n$ divides nx if and only if it divides n^2. Consequently, the number of x which yield positive integers y is precisely equal to the number of positive divisors of n^2. Indeed, for each positive divisor d of n^2, we take $x = n + d$.

In our case, $n = 100$, so $n^2 = 100^2 = 2^4{\cdot}5^4$. Thus the positive divisors of 100^2 are precisely the numbers of the form $2^a{\cdot}5^b$ with $a = 0, 1, 2, 3$ or 4 and with $b = 0, 1, 2, 3$ or 4. It follows that there are $5{\cdot}5 = 25$ divisors and hence there are 25 positive integers x that yield positive integers y.

2. In the figure, two circles that are tangent to each other are inscribed in a semicircle of radius 2. If the larger circle is tangent to the diameter of the semicircle at its midpoint C, find the radius of the smaller circle.



SOLUTION. If two circles are tangent to each other at a point T, then they have a common tangent line at T. Since the radii of the two circles at T are both perpendicular to this common line, they must be identical when extended. In particular, T and the centers of the two circles must be colinear.

In the figure, let A and B be the centers of the larger and smaller circles, respectively, and let x be the radius of the smaller circle. We draw the lines as indicated. First, note that C, A and the point of tangency E are colinear, so EC is a radius of the circle with center C and it is a diameter of the circle with center A. It follows that circle A has radius equal to 1. Next, since A, B and the point of tangency F are colinear, we have $AB = AF + FB = 1 + x$. Also, C, B and G are colinear, so $CB = CG - BG = 2 - x$. Finally, if H is the point of tangency of circle B and the diameter of the semicircle, then $\overline{BH}$ is perpendicular to $\overline{CH}$ and hence $DBHC$ is a rectangle. It follows that $DC = BH = x$, so $AD = AC - DC = 1 - x$.

By applying the Pythagorean Theorem to the right triangles $\triangle ADB$ and $\triangle CDB$, we obtain $(AB)^2 - (AD)^2 = (BD)^2 = (BC)^2 - (DC)^2$ or $(1+x)^2 - (1-x)^2 = (2-x)^2 - x^2$. This simplifies to $4x = 4 - 4x$, and consequently $x = 1/2$.

3. Let $\square$ be a binary operation defined on the set of nonnegative integers. (This means that if x and y are any two nonnegative integers, then $x \square y$ is a nonnegative integer determined by x and y.) Now suppose that the formula $(x \square y)(y \square z) = x \square z$ holds for all nonnegative integers x, y and z. If $23 \square 47 \neq 0$, compute $61 \square 89$.

SOLUTION. The answer is $61\square 89 = 1$. We will show more generally that if we are given any two nonnegative integers a and b with $a \square b \neq 0$, then $x \square y = 1$ for all x and y. To see this, note that the above formula yields $(a \square a)(a \square b) = a \square b$, and thus $a \square a = 1$ since $a \square b \neq 0$. Furthermore, we know that $(a \square x)(x \square y) = a \square y$, so $(a \square x)(x \square y)(y \square a) = (a \square y)(y \square a) = a \square a = 1$. But all the factors here are nonnegative integers, so they must all equal 1. In particular, we have $x \square y = 1$ for all choices of x and y.

4. I have 16 stones, each of which is either red, white or blue, and I place these stones on a 4×4 grid, with one stone at the center of each of the 16 boxes of the grid. Prove that there is at least one stone such that both its row and its column contain another stone of the same color.

SOLUTION. Consider the first row. Since there are three colors and four boxes, at least one color occurs twice, say red. Without loss of generality, we can assume that the two red stones occur in the first two positions as indicated by the R's in the left-most picture below. If there is an additional red stone in the first column, then we are done. Thus we can assume that the remaining three boxes of the first column contain only white or blue stones. Obviously one of these colors must occur at least twice. Say there are two white stones as indicated by the W's in the second picture below. If either of the two boxes in the second column directly to the right of the W's contains a red or white stone, then we have the desired configuration. Thus it suffices to assume that these boxes both contain blue stones, as is indicated by the B's in the third picture. Finally, if any of the remaining four boxes in the second or third row contains a white or blue stone, then again we are done. Thus we can assume that all four boxes contain red stones, thereby yielding the right-most picture and a square of red stones.

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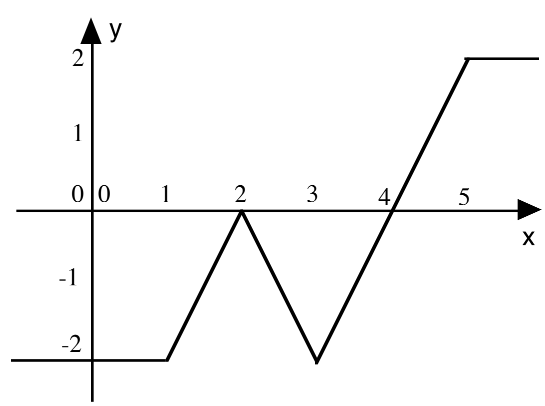
5. For which real numbers a does the equation $|x-1| - 2|x-2| + 2|x-3| - |x-5| = a$ have a unique solution. Here $|x|$ denotes the absolute value of x, defined by $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

SOLUTION. Since the graph of the curve $y = |x|$ is linear except when $x = 0$, the graph of the function $y = f(x) = |x-1| - 2|x-2| + 2|x-3| - |x-5|$ must also be linear except when one of the quantities within an absolute value sign is 0, that is except when $x = 1, 2, 3$ or 5. Note that for $x \geq 5$, we have $f(x) = (x-1) - 2(x-2) + 2(x-3) - (x-5) = 2$, so the line is horizontal here. Similarly, when $x \leq 1$, we have $f(x) = -(x-1) + 2(x-2) - 2(x-3) + (x-5) = -2$, and again the line is horizontal in this region. Since $f(2) = 0$ and $f(3) = -2$, it follows that the graph of $y = f(x)$ must be as pictured on the right.



Finally, a unique solution to the equation $f(x) = a$ occurs when the horizontal line $y = a$ meets the curve $y = f(x)$ at precisely one point. From the graph of $y = f(x)$, we see that this happens only for the values $0 < a < 2$.