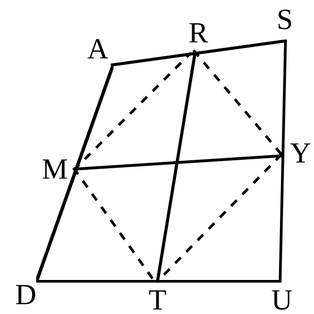
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2000-2001)

1. Let $a_1 = 14$, $a_2 = 144$ and in general, let a_n be the number $1444\cdots 4$, where there are n fours. Find all positive integers n such that a_n is a perfect square.

SOLUTION. We claim that $a_2 = 144 = 12^2$ and $a_3 = 1444 = 38^2$ are the only possible perfect squares. Since $a_1 = 14$ is not a square, it suffices to show that a_n is not a perfect square for $n \geq 4$. Suppose, by way of contradiction, that $a_n = b^2$ for some integer b. Since $n \geq 4$, we see that a_n ends with 4444, and since $10,000$ is divisible by 16, it follows that the remainder when a_n is divided by 16 is the same as the remainder when 4444 is divided by 16, namely 12. This implies, in particular, that b is even, but not divisible by 4. Thus $b = 2(2c + 1)$ for some integer c, and hence $b^2 = 4(4c^2 + 4c + 1) = 16(c^2 + c) + 4$. But then b^2 leaves a remainder of 4 when divided by 16, a contradiction, so a_n cannot be a perfect square when $n \geq 4$.

2. Let $ABCD$ be a quadrilateral. Suppose that points M, N, R, S, T and U are selected on the sides of the quadrilateral as shown, where M and N are the midpoints of sides $\overline{AD}$ and $\overline{BC}$, respectively; points R and S trisect side $\overline{AB}$ and points T and U trisect side $\overline{DC}$. Show that line $\overline{MN}$ bisects line segments $\overline{RT}$ and $\overline{SU}$.



SOLUTION. Let X be the midpoint of line segment $\overline{RT}$ and let Y be the midpoint of $\overline{SU}$. Consider the quadrilateral $ASUD$ and draw the lines $\overline{MR}$, $\overline{RY}$, $\overline{YT}$ and $\overline{TM}$, as indicated. We claim that $MRYT$ is a parallelogram. To see this, note that $AR = RS = AB/3$, so R is the midpoint of $\overline{AS}$. Thus since M is the midpoint of $\overline{AD}$, it follows that $\overline{MR}$ is parallel to $\overline{DS}$. Similarly, using the fact that Y is the midpoint of $\overline{SU}$, we have $\overline{YT}$ parallel to $\overline{DS}$, and hence $\overline{MR} \parallel \overline{YT}$. In the same way, we can argue that $\overline{RY} \parallel \overline{MT}$, and therefore $MRYT$ is indeed a parallelogram.

Now we know that the diagonals of a parallelogram bisect each other. By applying this to $\overline{MY}$ and $\overline{RT}$, we conclude that their point of intersection is the midpoint of $\overline{RT}$, namely X. In other words, we see that MXY is a straight line. In a similar manner, we can prove that XYN is a straight line, and hence X and Y are points on $\overline{MN}$. In particular, $\overline{MN}$ meets $\overline{RT}$ at its midpoint X and it meets $\overline{SU}$ at its midpoint Y.

3. For each integer n there is defined a certain integer n^*, depending on n. Suppose that $1^* = 1$ and that $a^*b^* = (a + b)^* + (a - b)^*$ for all integers a and b. Compute 100^*.

SOLUTION. Setting $a = 1$ and $b = 0$ in the given equation yields $1^*0^* = 1^* + 1^*$, so $1^* = 1$ implies that $0^* = 2$. Next, setting $b = 1$ and leaving a arbitrary yields $a^*1^* = (a+1)^* + (a-1)^*$, so $(a + 1)^* = a^* - (a - 1)^*$. In particular, $(a + 1)^*$ just depends on the star values of the preceding two numbers a and $a - 1$. Since $0^* = 2$ and $1^* = 1$, we get in turn $2^* = 1^* - 0^* = -1$, $3^* = 2^* - 1^* = -2, 4^* = 3^* - 2^* = -1, 5^* = 4^* - 3^* = 1, 6^* = 5^* - 4^* = 2$ and $7^* = 6^* - 5^* = 1$. Observe that $0^* = 6^*$ and that $1^* = 7^*$. Thus, since the star value of a number just depends upon the star values of the preceding two numbers, it follows that the pattern of star values must repeat in segments of length 6. We therefore conclude from $100 = 6 \cdot 16 + 4$ that $100^* = 4^* = -1$.

4. Let X be a set of positive integers with the property that for every nonempty finite subset Y of X, the average of all the numbers in Y is an integer. Show that there exists such a set X containing 1000 (different) numbers, but that it is impossible for X to be infinitely large.

SOLUTION. Let a be a positive integer that is divisible by all of the numbers b with $1 \leq b \leq 1000$. (For example, take $a = 1000!$.) Now let X be the set $\{a, 2a, 3a, \ldots, 1000a\}$, so that X has 1000 members. If Y is a nonempty subset of X, then the average of the members of Y is the sum s of the members of Y divided by the size b of the set Y. But s is a multiple of a, and a is a multiple of b, so the average s/b is an integer.

Now suppose X is an infinite set with the stated property and let n and m be members of X with $n > m$. Let Y be a subset of X consisting of n numbers, and choose Y so that it contains m but not n. Let s be the sum of the members of Y and note that s must be a multiple of n because s/n is the average of Y. Now create a new set Z by deleting m from Y and replacing it with n. The sum of the members of Z is $s - m + n$ and because the size of Z is n, it follows that $s - m + n$ is a multiple of n. Since we already know that s is a multiple of n, it follows that m must be a multiple of n. This is impossible, however, since $n > m > 0$.

5. Let a, b and c be positive numbers. Prove that

$$2(a^8 + b^8) \geq (a^3 + b^3)(a^5 + b^5)\,, \quad \text{and}$$
$$3(a^8 + b^8 + c^8) \geq (a^3 + b^3 + c^3)(a^5 + b^5 + c^5)\,.$$

SOLUTION. If we assume (as we may) that $a \geq b$, then we have

$$2(a^8 + b^8) - (a^3 + b^3)(a^5 + b^5) = a^8 + b^8 - a^3b^5 - b^3a^5 = (a^3 - b^3)(a^5 - b^5) \geq 0\,,$$

and this proves the first inequality.

Now apply the first inequality three times: to a and b, to a and c, and to b and c. If we add the three resulting inequalities, we get

$$\begin{aligned} 4a^8 + 4b^8 + 4c^8 &\geq (a^3 + b^3)(a^5 + b^5) + (a^3 + c^3)(a^5 + c^5) + (b^3 + c^3)(b^5 + c^5) \\ &= 2(a^8 + b^8 + c^8) + a^3b^5 + a^5b^3 + a^3c^5 + a^5c^3 + b^3c^5 + b^5c^3 \\ &= a^8 + b^8 + c^8 + (a^3 + b^3 + c^3)(a^5 + b^5 + c^5)\,. \end{aligned}$$

If we now subtract $a^3 + b^3 + c^3$ from both sides, we obtain the second inequality that we were asked to prove.