1. Suppose $\frac{a}{b} > \frac{x}{y} > \frac{c}{d}$ where $a, b, c, d, x$ and $y$ are non-negative integers. If $ad - bc = 1$, show that $x \geq a + c$ and $y \geq b + d$.

2. In $\triangle ABC$, point $X$ lies on side $BC$, one third of the way from $B$ to $C$. Similarly, $Y$ is chosen on $CA$, one third of the way from $C$ to $A$, and $Z$ lies on $AB$, one third of the way from $A$ to $B$. If the area of $\triangle ABC$ is 1 unit, find the area of $\triangle XYZ$.

3. Let $\square$ be a binary operation defined on the set of nonnegative integers. (This means that if $x$ and $y$ are any two nonnegative integers, then $x \square y$ is a nonnegative integer determined by $x$ and $y$.) Now suppose that (a) $(x + 1) \square 0 = (0 \square x) + 1$, (b) $0 \square (y + 1) = (y \square 0) + 1$, and (c) $(x + 1) \square (y + 1) = (x \square y) + 1$ are satisfied for all nonnegative integers $x$ and $y$. If $1100 \square 450 = 2000$, find $1723 \square 3421$ and prove that your answer is correct.

4. Sixteen numbers are put into the boxes of a four-by-four array so as to form a magic square. This means that the four row sums, the four column sums and the two diagonal sums are each equal to the same number $s$. Show that $s$ is also the sum of the four numbers in the corners of the array. (Do not assume that the sixteen numbers are the integers $1, 2, 3, \ldots, 16$.)

5. Consider polynomial equations of the form $x^3 + ax^2 + bx + 6 = 0$, where $a$ and $b$ are integers. Suppose that one of these equations has both $r$ and $-r$ as roots, where $r$ is a nonnegative real number. Find all possibilities for $r$. 

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on the problem set page. Remember that solutions usually require a proof or justification.