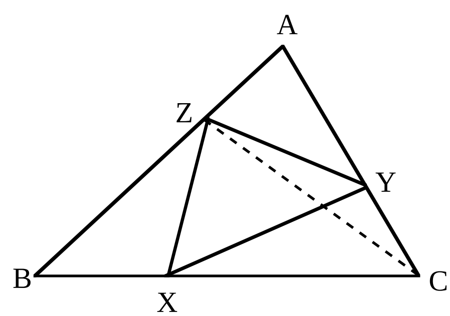
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2000-2001)

1. Suppose $\frac{a}{b} > \frac{x}{y} > \frac{c}{d}$ where a, b, c, d, x and y are nonnegative integers. If $ad - bc = 1$, show that $x \geq a + c$ and $y \geq b + d$.

SOLUTION. Note that $0 < \frac{a}{b} - \frac{x}{y} = \frac{ay - bx}{by}$, so $ay - bx = r$ is a positive integer. Similarly, $0 < \frac{x}{y} - \frac{c}{d} = \frac{dx - cy}{dy}$, so $dx - cy = s$ is also a positive integer. We can now use the fact that $ad - bc = 1$ to solve the simultaneous linear equations $-bx + ay = r$ and $dx - cy = s$ for x and y. Indeed, if we add c times the first equation to a times the second, we obtain $x = as + cr$, and if we add d times the first equation to b times the second, we get $y = bs + dr$. Since $r, s \geq 1$, we can then conclude that $x = as + cr \geq a + c$ and $y = bs + dr \geq b + d$.

2. In $\triangle ABC$, point X lies on side $\overline{BC}$, one third of the way from B to C. Similarly, Y is chosen on $\overline{CA}$, one third of the way from C to A, and Z lies on $\overline{AB}$, one third of the way from A to B. If the area of $\triangle ABC$ is 1 unit, find the area of $\triangle XYZ$.



SOLUTION. Draw the line $\overline{CZ}$ as indicated and note that $\text{Area}(\triangle BXZ) = (1/3)\,\text{Area}(\triangle BCZ)$ since these triangles have the same height to point Z and since their bases satisfy $BX = (1/3)\,BC$. Furthermore, $\text{Area}(\triangle BCZ) = (2/3)\,\text{Area}(\triangle BCA)$ since these triangles have the same height to point C and since their bases satisfy $ZB = (2/3)\,AB$. Combining this information, we see that $\text{Area}(\triangle BXZ) = (2/3)(1/3)\,\text{Area}(\triangle BCA) = 2/9$. Similarly $\text{Area}(\triangle CYX) = 2/9 = \text{Area}(\triangle AZY)$, and it follows that $\text{Area}(\triangle XYZ) = 1 - 3(2/9) = 1/3$.

3. Let $\square$ be a binary operation defined on the set of nonnegative integers. (This means that if x and y are any two nonnegative integers, then $x \,\square\, y$ is a nonnegative integer determined by x and y.) Now suppose that

(a) $(x + 1) \,\square\, 0 = (0 \,\square\, x) + 1$,
(b) $0 \,\square\, (y + 1) = (y \,\square\, 0) + 1$, and
(c) $(x + 1) \,\square\, (y + 1) = (x \,\square\, y) + 1$

are satisfied for all nonnegative integers x and y. If $1100 \square 450 = 2000$, find $1723 \square 3421$ and prove that your answer is correct.

SOLUTION. Write $A = 0 \,\square\, 0$, so that A is a nonnegative integer. Our first goal is to show that $(*)$ $x \,\square\, 0 = A + x = 0 \,\square\, x$ for every nonnegative integer x. Observe that $(*)$ is true when $x = 0$ by the definition of A. Now suppose we already know that $(*)$ holds for some number x. By (a), we have $(x + 1) \,\square\, 0 = (0 \,\square\, x) + 1 = (A + x) + 1 = A + (x + 1)$. Also (b) yields $0 \,\square\, (x + 1) = (x \,\square\, 0) + 1 = (A + x) + 1 = A + (x + 1)$, and we see that $(*)$ holds for $x + 1$. Since we know that $(*)$ is valid for $x = 0$, it follows that it is true for $x = 1$. Since we know that $(*)$ is valid for $x = 1$, it follows that it is true for $x = 2$. Continuing in this manner, we see that $(*)$ holds for every nonnegative integer x.

Now we consider $x \square y$, where x and y are both greater than 0. Suppose first that $y \geq x$. If we apply (c) a total of x times, we get

$$x \square y = 1 + (x-1) \square (y-1) = 2 + (x-2) \square (y-2) = \cdots = x + 0 \square (y-x)$$

and $(*)$ yields $x \square y = x + A + (y-x) = A + y$. Similarly, if $x \geq y$, then $x \square y = A + x$. Thus in all cases, $x \square y = A + \max\{x, y\}$, where $\max\{x, y\}$ is the larger of x and y. Finally, we are given $2000 = 1100 \square 450 = A + \max\{1100, 450\} = A + 1100$, so $A = 900$, and we deduce that $1723 \square 3421 = A + \max\{1723, 3421\} = 900 + 3421 = 4321$.

4. Sixteen numbers are put into the boxes of a four-by-four array so as to form a magic square. This means that the four row sums, the four column sums and the two diagonal sums are each equal to the same number s. Show that s is also the sum of the four numbers in the corners of the array. (Do not assume that the sixteen numbers are the integers $1, 2, 3, \ldots, 16$.)

SOLUTION. First observe that the sum of all 16 numbers is the sum of the four rows, which is $4s$. The sum of the middle two rows, the middle two columns and the two diagonals is $6s$ and this uses all of the numbers once except for the middle four, each of which is used three times. If we write m to denote the sum of the middle four numbers, we see that $6s$ is the sum of all the numbers plus an extra $2m$. This gives the equation $6s = 4s + 2m$, from which we deduce that $m = s$. Finally note that the sum of the two diagonals is $2s$, and this is the sum of the four middle boxes plus the sum of the four corners. Since the middle boxes sum to $m = s$, it follows that the corner boxes also sum to s, as wanted.

5. Consider polynomial equations of the form $x^3 + ax^2 + bx + 6 = 0$, where a and b are integers. Suppose that one of these equations has both r and $-r$ as roots, where r is a nonnegative real number. Find all possibilities for r.

SOLUTION. We are given $r^3 + ar^2 + br + 6 = 0$ and $(-r)^3 + a(-r)^2 + b(-r) + 6 = 0$. Adding these two equations yields $2ar^2 + 12 = 0$ and hence $ar^2 + 6 = 0$. Furthermore, by subtracting the original two equations, we obtain $2r^3 + 2br = 0$, so $r^2 + b = 0$ since it is clear that $r \neq 0$. Thus $6 = a(-r^2) = ab$ and, since a and b are integers, there are only a few possible choices for these numbers. In fact, since r is real and $b = -r^2$, we must have $b < 0$. Therefore we can have only $b = -1, -2, -3$ or -6 and hence $r = \sqrt{-b} = 1, \sqrt{2}, \sqrt{3}$ or $\sqrt{6}$.

Conversely, if r is one of these values, we can define $b = -r^2$ and $a = 6/b$. Then a and b are integers, and we observe that $r^2 + b = 0$ and $ar^2 + 6 = ar^2 + ab = a(r^2 + b) = 0$. Since $x^3 + ax^2 + bx + 6 = x(x^2 + b) + (ax^2 + 6)$, it follows that if we plug in either $x = r$ or $x = -r$ in this polynomial, then we get the value 0. In other words, both r and $-r$ are roots of the equation $x^3 + ax^2 + bx + 6 = 0$.