## WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

### SOLUTIONS TO PROBLEM SET III (2000-2001)

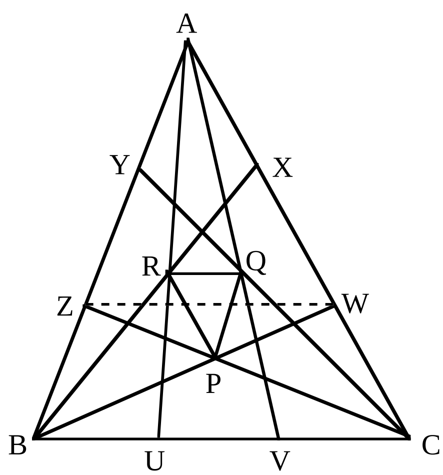
**1.** Let $m$ be a positive integer and suppose that $4m+1 = u^2+v^2$, where $u$ and $v$ are integers. Show that there exist integers $a$ and $b$ such that $m = \dfrac{a^2+a}{2} + \dfrac{b^2+b}{2}$.

**SOLUTION.** Note that $m = \dfrac{a^2+a}{2} + \dfrac{b^2+b}{2}$ is equivalent to

$$4m+1 = 2a^2+2a+2b^2+2b+1 = (a+b+1)^2 + (a-b)^2.$$

Now we know that $4m+1 = u^2+v^2$. Thus, if we can solve the equations $a+b+1 = u$ and $a-b = v$ for integers $a$ and $b$, then $m$ will have the appropriate form. Since the solutions are $a = \dfrac{u+v-1}{2}$ and $b = \dfrac{u-v-1}{2}$, it merely suffices to show that both numerators are even. But $4m+1$ is odd, so $4m+1 = u^2+v^2$ implies that one of $u$ or $v$ is even and the other is odd. Thus the numbers $u \pm v - 1$ are even, so $a$ and $b$ are indeed integers.

**2.** In $\triangle ABC$, the points $U$ and $V$ trisect side $\overline{BC}$, points $W$ and $X$ trisect side $\overline{AC}$, and points $Y$ and $Z$ trisect side $\overline{AB}$. Points $P$, $Q$ and $R$, as shown, are intersection points of lines joining vertices $A$, $B$ and $C$ to the trisection points on the opposite sides. Prove that the sides of $\triangle PQR$ are parallel to the sides of $\triangle ABC$.




**SOLUTION.** Draw the line $\overline{ZW}$, as indicated. We note that $AZ/AB = 2/3 = AW/AC$, and hence $\triangle AZW \sim \triangle ABC$ by SAS. Thus $\angle AZW = \angle ABC$ and $ZW/BC = 2/3$. It now follows that $\overline{ZW}$ is parallel to $\overline{BC}$, and thus $\triangle WZP \sim \triangle BCP$ since $\angle WZP = \angle BCP$ and $\angle ZWP = \angle CBP$. Consequently, $WP/PB = WZ/BC = 2/3$, and hence $BW = WP + PB = (2/3)PB + PB = (5/3)PB$. We now see that $BP/BW = 3/5$ and similar reasoning shows that $BR/BX = 3/5$. From this, it follows that $\triangle BRP \sim \triangle BXW$ by SAS, and thus $\angle BRP = \angle BXW$. We can now conclude that $\overline{RP}$ is parallel to $\overline{AC}$ and similarly, the other two sides of $\triangle PQR$ are parallel to the other two sides of $\triangle ABC$.

**3.** Let $a$, $b$ and $c$ be positive numbers. Show that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a+b+c.$$

**SOLUTION.** Notice that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{a^2-ab+b^2}{b} + \frac{b^2-bc+c^2}{c} + \frac{c^2-ca+a^2}{a}$$

since $(-a+b)+(-b+c)+(-c+a)=0$. Furthermore, $a^2-ab+b^2 = ab+(a-b)^2 \geq ab$, and similarly $b^2-bc+c^2 \geq bc$ and $c^2-ca+a^2 \geq ca$. Thus, since $a$, $b$ and $c$ are positive numbers, we get

$$\frac{a^2-ab+b^2}{b}+\frac{b^2-bc+c^2}{c}+\frac{c^2-ca+a^2}{a} \geq \frac{ab}{b}+\frac{bc}{c}+\frac{ca}{a} = a+b+c.$$

**4.** (New Year's Problem) How many pairs of positive integers $a$ and $b$ are there such that $a < b$ and $\frac{1}{a}+\frac{1}{b}=\frac{1}{2001}$ ?

**SOLUTION.** Certainly we must have $a > 2001$, so we can write $a = 2001+k$ for some positive integer $k$. Then

$$\frac{1}{b}=\frac{1}{2001}-\frac{1}{2001+k}=\frac{k}{2001\cdot(2001+k)},$$

so

$$b=\frac{2001\cdot(2001+k)}{k}=2001+\frac{2001^2}{k}.$$

Since $b$ is an integer larger than $a = 2001+k$, it follows that $k$ must be a divisor of $2001^2$ that is less than the square root of $2001^2$. Now $2001 = 3\cdot 23\cdot 29$, so $2001^2 = 3^2\cdot 23^2\cdot 29^2$ has $3\cdot 3\cdot 3 = 27$ positive integer divisors. Furthermore, if $s$ is any such divisor with $s < 2001$, then $b = 2001^2/s$ is a divisor with $b > 2001$. Thus, precisely 13 of these divisors are less than 2001.

**5.** Let $S$ be a set of 100 positive integers, each less than 200. Show that there exists a nonempty subset $T$ of $S$ such that the product of all of the numbers in $T$ is a perfect square.

**SOLUTION.** For each nonempty subset $Z$ of $S$, let $\Pi(Z)$ denote the product of all elements of $Z$ and let $m_Z$ denote the square-free part of this product. In other words, $\Pi(Z) = c^2 m_Z$ where $c^2$ is the largest square integer dividing the product. Thus, each $m_Z$ is a product of distinct primes, each less than 200, and since there are fewer than 100 such primes, we see that there are at most $2^{99}$ possibilities for $m_Z$. Now the number of possible $Z$ is the number of nonempty subsets of $S$, namely $2^{100}-1$. Thus, since $2^{100}-1 > 2^{99}$, there must exist two distinct nonempty subsets $X$ and $Y$ of $S$ such that $m_X = m_Y$. If $\Pi(X) = a^2 m_X$ and $\Pi(Y) = b^2 m_Y = b^2 m_X$, then $\Pi(X)\cdot\Pi(Y) = a^2 m_X\cdot b^2 m_X$ is a perfect square. Now let $T$ be the set of all elements in either $X$ or $Y$ but not in both. As usual, $T$ is called the symmetric difference of $X$ and $Y$. Since $X \neq Y$, we see that $T$ is not empty. Furthermore, let $X \cap Y$ denote the set of elements contained in both $X$ and $Y$. Then each number in $X \cap Y$ occurs twice in the product $\Pi(X)\cdot\Pi(Y)$, so we have $\Pi(X)\cdot\Pi(Y) = \Pi(T)\cdot\big(\Pi(X\cap Y)\big)^2$. Since $\Pi(X)\cdot\Pi(Y)$ is a perfect square, so is $\Pi(T)$.