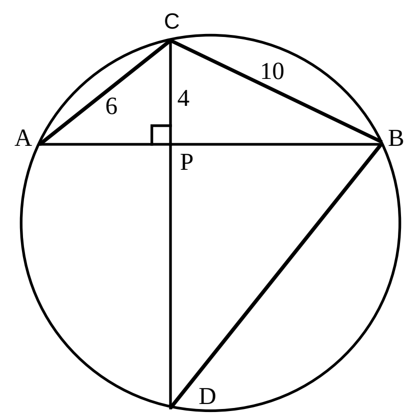
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2000-2001)

1. Two race cars are traveling around a track at a constant speed, each of them taking one hour to complete one lap. One car starts at midnight and the other starts at 5:20 AM that morning. At what times after the second car starts is it true that the number of laps completed by the first car is exactly double the number of laps completed by the second car? At what times is it triple the number?

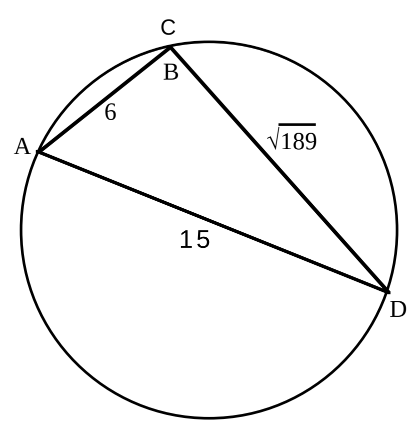
SOLUTION. Let $T \geq 6$ be an integer and let us consider a time t after midnight with $T{:}00 \leq t < T{:}20$. Then the first car has completed T laps while the second has completed $T - 6$. In particular, if $T = 2(T-6)$ then $T = 12$, while $T = 3(T-6)$ yields $T = 9$. On the other hand, if $T \geq 5$ and $T{:}20 \leq t < (T+1){:}00$, then the first car has again completed T laps, while the second has now done $T - 5$. If $T = 2(T-5)$ then $T = 10$, and if $T = 3(T-5)$ then $T = 15/2$ is not an integer. Thus we see that the number of laps completed by the first car is exactly double the number of laps completed by the second car when 10:20 AM $\leq t <$ 11:00 AM and when 12:00 noon $\leq t <$ 12:20 PM. On the other hand, the number of laps completed by the first car is exactly triple the number of laps completed by the second car when 9:00 AM $\leq t <$ 9:20 AM.

2. In the circle, chord $\overline{AB}$ is drawn and a point C on the circle is selected so that the perpendicular distance to the chord is 4 units, as indicated. If $CA = 6$ and $CB = 10$, find the diameter of the circle.



SOLUTION. Let P denote the point of intersection of the chord $\overline{AB}$ with the perpendicular line from C, and extend $\overline{CP}$ until it meets the circle at D. Since $\triangle APC$ is a right triangle, we have $AP = \sqrt{20}$. Similarly, $\triangle BPC$ is right, so $BP = \sqrt{84}$. Next, since $\overline{AB}$ and $\overline{CD}$ are chords meeting at P, we know that $AP \cdot BP = CP \cdot DP$ and hence $DP = \sqrt{105}$. We conclude from right triangle $\triangle BPD$ that $BD = \sqrt{189}$. Since $\angle APC = 90^\circ$, we have $\overset{\frown}{AC} + \overset{\frown}{BD} = 180^\circ$.

Let us move the chord $\overline{BD}$, as indicated in the second diagram, so that the points B and C become one. Then since $\overset{\frown}{AC} + \overset{\frown}{BD} = 180^\circ$, it follows that the new chord $\overline{AD}$ is a diameter of the circle and that $\angle ACD = 90^\circ$. The Pythagorean Theorem now yields $AD^2 = AC^2 + BD^2 = 225$, so the diameter has length 15.



An alternative argument for this problem uses the fact that if $\triangle ABC$ is inscribed in a circle of radius R, then $4KR = abc$. Here K is the area of the triangle and a, b and c are the lengths of the three sides. In our case, $a = BC = 10$, $b = AC = 6$, and since $\overline{CP}$ is an altitude of length 4 to side c, we have $K = 2c$. Thus $abc = 4KR = 8cR$, so $2R = ab/4 = 60/4 = 15$.

3. Exactly how many numbers n in the range $1 \leq n \leq 1000$ can be written as $n = r^2 - s^2$ for positive integers r and s.

SOLUTION. Let a be a nonnegative integer and observe that $(a+1)^2 - a^2 = 2a+1$. This shows that every odd number $n = 2a+1$ can be written as a difference of two squares. But note that $n = 1$ implies that $a = 0$ and $s = a$ is not positive. Indeed, if $1 = r^2 - s^2$, then it is easy to see that s must be 0. Thus, all odd positive numbers n can be written as a difference of two positive squares except $n = 1$.

Next observe that $(a+1)^2 - (a-1)^2 = 4a$, so every number $n = 4a$ divisible by 4 can be written as a difference of two squares. But again note that $n = 4$ implies that $a = 1$ and $s = a - 1 = 0$. Indeed, if $4 = r^2 - s^2$, then it is easy to see that s must be 0. Thus, all positive numbers n divisible by 4 can be written as a difference of two positive squares except $n = 4$.

Finally, if n is twice an odd number and if $n = r^2 - s^2$, then r and s are either both even or both odd. But then $n = (r-s)(r+s)$ is divisible by 4, a contradiction. Thus no such number is a difference of two squares. Since there are 500 odd numbers n with $1 \le n \le 1000$ and since there are 250 numbers divisible by 4 in this range, we conclude that there are precisely $500 + 250 - 2 = 748$ numbers in this range which can be written as a difference of two positive squares. The term -2 in this formula occurs since we cannot include $n = 1$ or $n = 4$.

4. Show that

$$\sqrt[3]{w^3 + x^3 + y^3 + z^3} \le \sqrt{w^2 + x^2 + y^2 + z^2}$$

for all choices of real numbers w, x, y and z.

SOLUTION. Let $\sqrt{w^2 + x^2 + y^2 + z^2} = t$. If $t = 0$, then $w = x = y = z = 0$ so the inequality is certainly satisfied. Now suppose $t > 0$ and observe that $(w/t)^2 + (x/t)^2 + (y/t)^2 + (z/t)^2 = 1$. Since each of these four summands is nonnegative, we see that each of them is ≤ 1. Now if a is a real number with $a^2 \le 1$, then it is easy to see that $a^3 \le a^2$. Indeed, this is clear is $a \le 0$. On the other hand, if $a > 0$, then $a \le 1$ so $a^3 = a^2 \cdot a \le a^2 \cdot 1 = a^2$. Thus

$$(w/t)^3 + (x/t)^3 + (y/t)^3 + (z/t)^3 \le (w/t)^2 + (x/t)^2 + (y/t)^2 + (z/t)^2 = 1$$

and since $t > 0$, this yields $w^3 + x^3 + y^3 + z^3 \le t^3$. Consequently, $\sqrt[3]{w^3 + x^3 + y^3 + z^3} \le t$.

5. Let us say that a positive integer n is *obtainable* if there exist integers x and y such that $n = 2x^2 + 3y^2$. If n is obtainable, prove that $7n$ is also obtainable.

SOLUTION. Let $n = 2x^2 + 3y^2$ be an obtainable number. We consider expressions of the form $2(ax+by)^2 + 3(cx+dy)^2$ with a, b, c and d integers in the hope of finding the number $7n$. In other words, we want

$$\begin{aligned} 14x^2 + 21y^2 = 7n &= 2(ax+by)^2 + 3(cx+dy)^2 \\ &= (2a^2 + 3c^2)x^2 + (4ab + 6cd)xy + (2b^2 + 3d^2)y^2. \end{aligned}$$

In fact, we want this equation to hold for all integers x and y, and this requires the coefficients of x^2, xy and y^2 to match on both sides. Thus we need $2a^2 + 3c^2 = 14$, $4ab + 6cd = 0$ and $2b^2 + 3d^2 = 21$. In particular, a, b, c and d must be fairly small integers, and by experimentation we discover one solution to be $a = 1$, $b = -3$, $c = 2$ and $d = 1$. Thus $7n = 2(x-3y)^2 + 3(2x+y)^2$ is indeed obtainable.