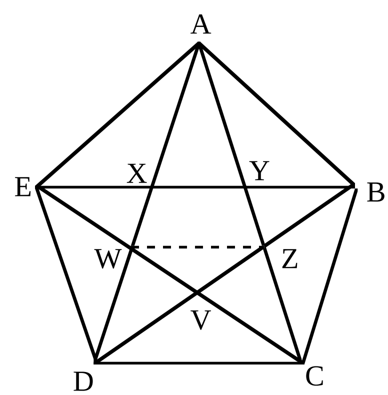
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2000-2001)

1. Find all positive integers n such that $n^4 + n^3 + 1$ is a perfect square.

SOLUTION. Suppose n^4+n^3+1 is a perfect square. Since $n^4+n^3+1 > n^4 = (n^2)^2$, we see that $n^4+n^3+1 = (n^2+k)^2 = n^4+2kn^2+k^2$ for some positive integer k. Thus $n^2(n-2k) = k^2-1 \geq 0$. In particular, n^2 divides $k^2 - 1$, so either $k^2 = 1$ or $n^2 \leq k^2 - 1$. If $k^2 = 1$, then $k = 1$ and $n^2(n-2) = 0$, so $n = 2$ is a solution. Indeed, $2^4 + 2^3 + 1 = 25 = 5^2$. On the other hand, if $k \neq 1$, then $k^2 > k^2 - 1 \geq n^2$, so $k > n$. But then $n - 2k$ is negative and this contradicts the fact that $n^2(n-2k) = k^2 - 1 \geq 0$.

2. Let us say that a pentagon $ABCDE$ is *special* if each diagonal is parallel to its opposite side. In other word, the pentagon in the figure is special if and only if $EB\|DC$, $AC\|ED$, $BD\|AE$, $CE\|BA$ and $DA\|CB$. If $ABCDE$ is special, prove that the areas of the five triangles $\triangle AXY$, $\triangle BYZ$, $\triangle CZV$, $\triangle DVW$ and $\triangle EWX$ forming the points of the star are equal, and that $VWXYZ$ is a special pentagon.



SOLUTION. First observe that triangles EDC and BDC have the same base $\overline{DC}$ and the same altitude since $EB\|DC$. Thus these triangles have the same area. Similarly $\triangle DCB$ and $\triangle ACB$ have the same area. Continuing in this manner, we conclude that the five triangles $\triangle EDC$, $\triangle DCB$, $\triangle CBA$, $\triangle BAE$ and $\triangle AED$ all have the same area, say α.

Next observe that $EB\|DC$, $AC\|ED$ and $DA\|CB$ imply that $EDCY$ and $XDCB$ are parallelograms. Thus $EY = DC = XB$ and $EX = EY - XY = XB - XY = YB$. In particular, $\triangle AEX$ and $\triangle AYB$ have the same base $EX = YB$ and the same altitude, so they have the same area. Similarly, $\triangle BAY$ and $\triangle BZC$ have the same area. Continuing in this manner, we conclude that the five triangles $\triangle AEX$, $\triangle BAY$, $\triangle CBZ$, $\triangle DCV$ and $\triangle EDW$ all have the same area, say β. It follows that each of the star triangles, $\triangle AXY$, $\triangle BYZ$, $\triangle CZV$, $\triangle DVW$ and $\triangle EWX$, has area $\alpha - 2\beta$, so all these areas are equal.

Finally, note that $\triangle DCW$ and $\triangle DCZ$ have the same base DC and the same area $\alpha - \beta$. Thus, they must have the same altitude, and this implies that $WZ\|DC$. But $DC\|EB$, so $WZ\|XY$. The same argument works for the other four diagonals of $VWXYZ$ and shows that $VWXYZ$ is indeed a special pentagon.

3. Let A be the sum of ten positive real numbers and let B be the sum of the reciprocals of these ten numbers. What is the smallest possible value for AB?

SOLUTION. Let the ten positive real numbers be $x_1, x_2, \ldots, x_{10}$. Then $A = x_1 + x_2 + \cdots + x_{10}$, $B = 1/x_1 + 1/x_2 + \cdots + 1/x_{10}$, and hence AB is a sum of $10 \times 10 = 100$ terms. Ten of these terms are of the form $x_i \cdot (1/x_i) = 1$, for some subscript i. The other 90 terms consist of the 45 pairs $\{x_i/x_j, x_j/x_i\}$ with $1 \leq i < j \leq 10$, and we note that $(x_i/x_j + x_j/x_i)^2 = 4 + (x_i/x_j - x_j/x_i)^2 \geq 4$.

Thus $x_i/x_j + x_j/x_i \geq 2$ and $AB \geq 10{\cdot}1 + 45{\cdot}2 = 100$. Of course, 100 does occur as a product AB by taking $x_1 = x_2 = \cdots = x_{10}$, so 100 is the smallest possible value for AB.

4. A magic money machine does the following. If you put in a penny, out comes a dime; if you put in a dime, out come a penny and a quarter; and if you put in a quarter, out come two dimes. Starting with one dime, I play with the machine for a while. When I count my money, I find that I have exactly 100 pennies, along with other coins. What is the smallest total amount of money I could possibly have?

SOLUTION. Suppose that over the time I played with the machine, I inserted a total of p pennies, d dimes and q quarters and assume that at the end of my play I have P pennies, D dimes and Q quarters. We know, in fact, that $P = 100$. Each time I inserted a dime, my supply of pennies went up by 1 and each time I inserted a penny, my supply of pennies went down by 1. Of course, inserting quarters had no effect on my supply of pennies. Since I started with no pennies, we see that $P = d - p$. Each time I inserted a dime, my supply of quarters went up by 1 and each time I inserted a quarter, my supply of quarters went down by 1. Since inserting pennies had no effect on my supply of quarters and since I started with no quarters, it follows that $Q = d - q$. Finally, we compute D by observing that each time I inserted a penny, my supply of dimes increased by 1; each time I inserted a dime it decreased by 1; and each time I inserted a quarter, my supply of dimes increased by 2. Since I started with 1 dime, we see that $D = 1 + p - d + 2q$.

From the equations we just derived, we see that $D + 2Q = 1 + d + p \geq 1 + d - p = 1 + P$, where the inequality holds because $p \geq 0$. If we let M denote the amount of money I have (in cents), we see that

$$M = P + 10D + 25Q \geq P + 10(D + 2Q) \geq P + 10(P + 1) = 11P + 10 = 1110\,.$$

In other words, I must have a total of at least \$11.10.

To see that it really is possible to end up with exactly \$11.10, observe that the net effect of inserting one dime and then one quarter is to add one dime and one penny to the original supply of money. Starting with one dime and alternately inserting dimes and quarters for a total of 100 dimes and 100 quarters inserted, the effect is to add to my original dime exactly 100 extra dimes and 100 extra pennies. After doing this, I have exactly 100 pennies and 101 dimes, and this gives $M = \$11.10$, as required.

5. Find all pairs of nonnegative integers x and y such that $y^2(x + 1) = 1576 + x^2$.

SOLUTION. Suppose x and y are nonnegative integers with $y^2(x + 1) = 1576 + x^2$. Then $y^2(x+1) = 1577 + (x^2 - 1) = 1577 + (x-1)(x+1)$, so $(y^2 - x + 1)(x + 1) = 1577 = 19{\cdot}83$. In particular, $x + 1$ is a positive divisor of $19{\cdot}83$ with quotient $y^2 - x + 1$. If $x + 1 = 1$, then $x = 0$ and $y^2 = 1576$; if $x + 1 = 19$, then $x = 18$ and $y^2 = 100$; if $x + 1 = 83$, then $x = 82$ and $y^2 = 100$; and if $x + 1 = 1577$, then $x = 1576$ and $y^2 = 1576$. Since $\sqrt{1576} \approx 39.7$ is not an integer, the two possibilities are $x = 18$, $y = 10$ and $x = 82$, $y = 10$.